Four-valued Logics for Paraconsistent Reasoning

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- Semantic Web Reasoning
- Paraconsistent Reasoning
- Four-valued Approach

Semantic Web

- Universal medium for information exchange
- Enables machines to
  - Interpret information
  - Perform reasoning tasks
- Extension of reasoning capabilities
  - Usage of information from diverse sources
  - Reasoning on the joined information

Different Viewpoints

- **Viewpoint of a German:**
  - **"The writer Franz Kafka is German."**
- **Viewpoint of a Czech:**
  - **"The writer Franz Kafka is Czech."**
- **Viewpoint of a historian:**
  - **"Franz Kafka is AustroHungarian."**
  - **"Franz Kafka is not German."**
  - **"Franz Kafka is not Czech."**

Contradicting Information

- Joining information may cause inconsistencies
- Formalized example
  - $\Sigma_1 = \{$*German(FranzKafka)*, *Writer(FranzKafka)* *German(ThomasMann)*, *Writer(ThomasMann)*$\}$
  - $\Sigma_2 = \{$*Czech(FranzKafka)*, *Writer(FranzKafka)* *Czech(MilanKundera)*, *Writer(MilanKundera)*$\}$
  - $\Sigma_3 = \{$*Writer(ArthurSchnitzler)*, *AustroHungarian(FranzKafka)*, $\neg$*German(FranzKafka)*, $\neg$*Czech(FranzKafka)*$\}$
  - $\Sigma_4 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$

Problem of Inconsistencies – the Principle of Explosion

- **Rule of classical logic: *Anything follows from a contradiction***
  - $\Sigma_4 \models$ ***Irish(JoséCela)*** **since** Σ_4 **has no classical model**
- Σ_4 **becomes trivial and we lose all information**
- **However, valuable information exist, e.g., *Writer(FranzKafka)***
- **Desire to draw conclusions from consistent information**
- **Allowing** Σ_4 **to have a model avoids explosion**

What is Paraconsistency?

- A set of formulas Σ is *paraconsistent* iff
  - it is inconsistent, i.e.,

    we find for a formula F that: $\Sigma \models F$

    and for the negation of F that: $\Sigma \models \neg F$

  - but non-trivial, i.e.,

    Σ has a model

What is a Paraconsistent Logic?

Let $\langle \mathcal{L}, \models \rangle$ be a logic, $\mathcal{L}$ its set of formulas, $\models$ its entailment relation

- $\langle \mathcal{L}, \models \rangle$ is *paraconsistent* iff
  it allows paraconsistent sets of formulas

- $\langle \mathcal{L}, \models \rangle$ is *fully paraconsistent* iff
  all subsets of $\mathcal{L}$ are non-trivial

Our Objective

- Join sets of formulas by avoiding trivialization
  - A trivial set of formulas leads to explosion
  - Even paraconsistent logics may explode
- Obtain a logic that cannot explode
  - A fully paraconsistent logic
- Perform reasoning tasks on inconsistent information
  - Information is specified in classical logic
  - Interpret information in a paraconsistent sense

Proceeding in Thesis

- **Development of a paraconsistent first-order logic**
- **Definition of a semantic mapping to description logics**
- **Elaboration of paraconsistent description logics corresponding to $\mathcal{ALC}$ and $\mathcal{SHIQ}$**
- **Implementation of paraconsistent DL-Reasoning**

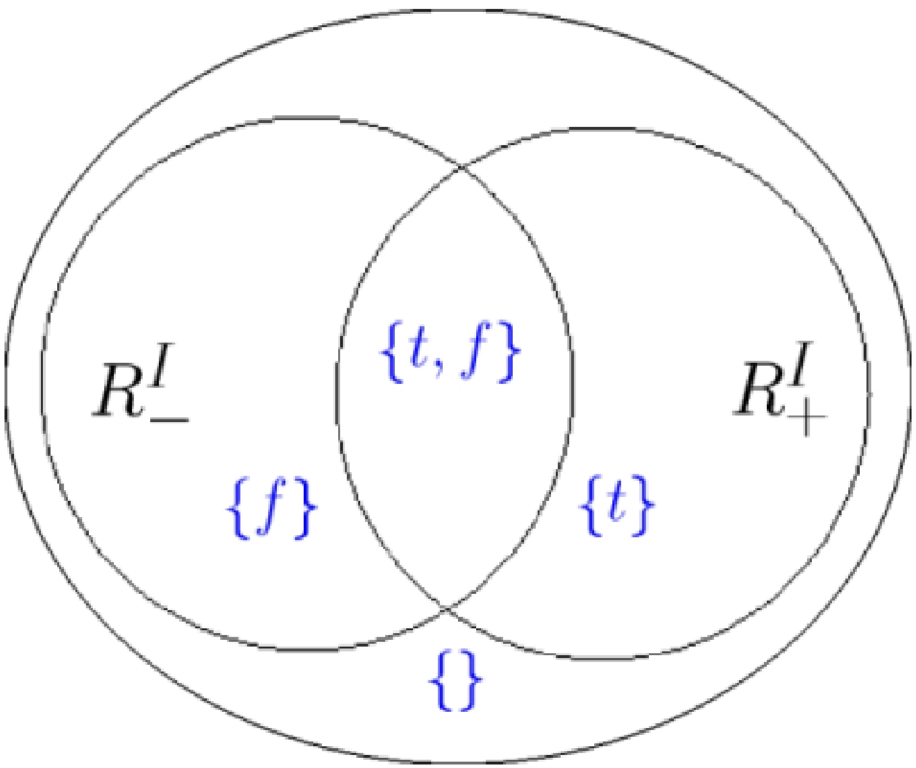
This presentation covers

- Paraconsistent first-order logic
- Reasoning in our paraconsistent logic
- Discussion of the realization of:
  - Subsumption, denoted by $\sqsubseteq$
  - Equivalence, denoted by $\equiv$

How to obtain Paraconsistency

- A logic allowing formulas to be *true* and *false* at the same time
- Intuitively providing additional truth values
- Multi-valued logics, e.g., four-valued logic (Nuel Belnap)
- Four-valued logics are a generalization of two-valued logics
- Formulas of classical logic are formulas of four-valued logic too

Idea of Four-valued Logics



An interpretation assigns to a relation symbol R two sets R_+^I and R_-^I

The truth value of $R(\vec{a})$ depends on the set membership of $\vec{a}^I$

$R(\vec{a})^I = \{\}$ iff $\vec{a}^I \notin R_+^I$ and $\vec{a}^I \notin R_-^I$

$R(\vec{a})^I = \{f\}$ iff $\vec{a}^I \notin R_+^I$ and $\vec{a}^I \in R_-^I$

$R(\vec{a})^I = \{t\}$ iff $\vec{a}^I \in R_+^I$ and $\vec{a}^I \notin R_-^I$

$R(\vec{a})^I = \{t, f\}$ iff $\vec{a}^I \in R_+^I$ and $\vec{a}^I \in R_-^I$

Logical Consequence in Four-valued Logics (I)

- **Definition of entailment relations $\models_t$ and $\models_f$**
  - $I, \sigma \models_t R(\vec{x})$ **iff** $\vec{x}^{I,\sigma} \in R_+^I$
  - $I, \sigma \models_f R(\vec{x})$ **iff** $\vec{x}^{I,\sigma} \in R_-^I$
  - $I, \sigma \models_t \neg F$ **iff** $I, \sigma \models_f F$
  - $I, \sigma \models_f \neg F$ **iff** $I, \sigma \models_t F$
  - $I, \sigma \models_t F \wedge G$ **iff** $I, \sigma \models_t F$ **and** $I, \sigma \models_t G$
  - $I, \sigma \models_f F \wedge G$ **iff** $I, \sigma \models_f F$ **or** $I, \sigma \models_f G$
  - $\ldots$

Logical Consequence in Four-valued Logics (II)

- ***Satisfiable*** **is a formula F iff $I, \sigma \models_t F$ holds for some I, σ**
  - **$\{t\}$ and $\{t, f\}$ are both *designated* truth values**
- **I is a *model* of F iff $I, \sigma \models_t F$ holds for all σ**

Logical Consequence in Four-valued Logics (III)

- **A set of formulas Γ *entails* a formula F iff all interpretations that are models of Γ are models of F**
  - **We write $\Gamma \models_t F$**
- **Formulas F and G are *semantically equivalent* iff $F^{I,\sigma} = G^{I,\sigma}$**
  - **We write $F \equiv G$**

Benefit of Four-valued Logics

- **A formula can be both *true* and *false* at the same time**
  - **Inconsistencies do not trivialize a set of formulas**
    $F^{I,\sigma} = \{t, f\}$ **and** $G^{I,\sigma} = \{f\}$ **but** $F \land \neg F \not\models_t G$
- **Full paraconsistency w.r.t. conjunction, disjunction and negation**
- **Intuitive calculation with conjunction, disjunction and negation**
- **Use of formulas of classical first-order logic**

Difficulties in Four-valued Logics

- Lack of four-valued reasoning systems
- Use of a truth or a false formula
- Definition of an implication connective
- Definition of an equivalence connective
- Definition of an equality relation

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Lack of four-valued reasoning systems

- We transform formulas F into negation normal form F'
- Function λ maps formula F' of four-valued first-order logic to a formula of a negation-free classical first-order logic

$$\lambda(F') \stackrel{\textit{def}}{=} \begin{cases} R_+(\vec{x}) & \text{for positive literals } R(\vec{x}), \\ R_-(\vec{x}) & \text{for negative literals } \neg R(\vec{x}), \\ \lambda(G) \circ \lambda(H) & \text{for two formulas } G \text{ and } H \\ & \text{with } \circ \in \{\wedge, \vee\}, \\ (Qx)\lambda(F) & \text{for a formula } F \text{ with } Q \in \{\forall, \exists\}. \end{cases}$$

- Application of reasoning systems for classical logic on $\lambda(F')$

Example Mapping to negation-free Classical Logic

- Translation of Σ_4 in negation normal form
  - $\Sigma_4' = \{$*German(FranzKafka)*, *Writer(FranzKafka)*, $\neg$*German(FranzKafka)*, $\ldots\}$
- Mapping of Σ_4' to negation free classical first-order logic
  - $\lambda(\Sigma_4') = \{$*German*$_+$*(FranzKafka)*, *Writer*$_+$*(FranzKafka)*, *German*$_-$*(FranzKafka)*, $\ldots\}$

Reasoning with standard reasoning systems

- $\lambda(\Sigma_4')$ does not explode and
  - {*German*$_+$*(FranzKafka), Writer*$_+$*(FranzKafka), German*$_-$*(FranzKafka)*, ...}
- Hence, $\Sigma_4 \models_t$ *Writer(FranzKafka)*
- and $\Sigma_4 \not\models_t$ *Irish(JoséCela)*

Difficulties in Four-valued Logics

- Lack of four-valued reasoning systems
- **Use of a truth or a false formula**
- Definition of an implication connective
- Definition of an equivalence connective
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The truth and the false formula $\top$ and $\bot$

- Description logics require a *top* and *bottom* concept
- Correspondence to top and false formula in first-order logic
- Intended semantics of $\top$ is to be always true and of $\bot$ to be always false
- Allowing $\top$ or $\bot$ makes the logic trivializable:

$$\neg\top \equiv \bot \rightsquigarrow \bot \vee \boldsymbol{F} \equiv \boldsymbol{F}$$

The truth and the false formula $\top$ and $\bot$

- Full paraconsistency cannot be obtained
  - A set of formulas containing $\bot$ is trivial
- Joining non-trivial sets of formulas yields a non-trivial set
- Easy to check for occurrences of $\top$ and $\bot$
- Careful use of $\top$ and $\bot$ may be allowed

Difficulties in Four-valued Logics

- Lack of four-valued reasoning systems
- Use of a truth or a false formula
- **Definition of an implication connective**
- Definition of an equivalence connective
- Definition of an equality relation

Requirement for Implication Connective

- Concept subsumption is basic scheme in DLs
  - $(D \sqsubseteq C)$ corresponds to $(\forall x)(D(x) \rightarrow C(x))$
- Giving semantics of DL by first-order logic relies on implication
  - $(\forall R.C)$ corresponds to $(\forall y)(R(x,y) \rightarrow C(y))$
  - $(\leq 3R.C)$ corresponds to $(\forall y_1, y_2, y_3, y_4)(\bigwedge_{i=1}^{4}(R(x,y_i) \wedge C(y_i)) \rightarrow \bigvee_{i,j} y_i \approx y_j)$

Definition of an Implication Connective (I)

- **Requirements for an implication connective**

1. **Full paraconsistency has to be preserved**
   - **to avoid that our logic becomes trivializable**
2. **The deduction theorem has to be satisfied**
   - **to express consequence relations as formulas**
3. **It should be expressible by $\wedge$, $\vee$, $\neg$**
   - **to map formulas to classical logic**

Definition of an Implication Connective (II)

- **Item 2 and item 3 are mutually exclusive**
- **We drop the third requirement**
- **Fortunately, transformation in classical logic is possible**
  - $R(x) \longrightarrow S(y)$ **is mapped to** $\neg R_{+}(x) \vee S_{+}(y)$

Difficulties in Four-valued Logics

- Lack of four-valued reasoning systems
- Use of a truth or a false formula
- Definition of an implication connective
- **Definition of an equivalence connective**
- Definition of an equality relation

Definition of an Equivalence Connective (I)

- **Common definition:** $F \leftrightarrow G \stackrel{\text{def}}{=} (F \rightarrow G) \wedge (G \rightarrow F)$
- **Corresponding truth table**

$\leftrightarrow$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\{\}$	$\{t\}$	$\{t\}$	$\{\}$	$\{\}$
$\{f\}$	$\{t\}$	$\{t,f\}$	$\{\}$	$\{f\}$
$\{t\}$	$\{\}$	$\{\}$	$\{t\}$	$\{t\}$
$\{t,f\}$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$

- **Equivalence not properly described by $\leftrightarrow$**

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$\{\}$	$\{t\}$	$\{t\}$	$\{\}$	$\{\}$
$\{f\}$	$\{t\}$	$\{t,f\}$	$\{\}$	$\{f\}$
$\{t\}$	$\{\}$	$\{\}$	$\{t\}$	$\{t\}$
$\{t,f\}$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$

- **Equivalence not properly described by $\leftrightarrow$**

Definition of an Equivalence Connective (II)

- **Definition of $\Leftrightarrow$ satisfies intuitive equivalence semantics**
- **Corresponding truth table**

$\Leftrightarrow$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\{\}$	$\{t\}$	$\{f\}$	$\{f\}$	$\{f\}$
$\{f\}$	$\{f\}$	$\{t\}$	$\{f\}$	$\{f\}$
$\{t\}$	$\{f\}$	$\{f\}$	$\{t\}$	$\{f\}$
$\{t,f\}$	$\{f\}$	$\{f\}$	$\{f\}$	$\{t,f\}$

- **Problematic in DL since equivalence often described by subsumption**

Subsumption and Equivalence in $\mathcal{ALC}4$

- **Let A and B be two concepts**
- $A \sqsubseteq B$ **if and only if**
  - $A_+ \subseteq B_+$
- $A \equiv B$ **if and only if**
  - $(A_+ = B_+)$ **and** $(A_- = B_-)$
  - $(A_+ \subseteq B_+)$ **and** $(B_+ \subseteq A_+)$ **and** $(A_- \subseteq B_-)$ **and** $(B_- \subseteq A_-)$

Difficulties in Four-valued Logics

- Lack of four-valued reasoning systems
- Use of a truth or a false formula
- Definition of an implication connective
- Definition of an equivalence connective
- **Definition of an equality relation**

The Equality Relation $\approx$

- Requirement for equality in description logic $\mathcal{SHIQ}$
  - E.g. number restriction relies on equality
- Intended semantics of $a \approx b$ is: $a^{I,\sigma} = b^{I,\sigma}$
- Such a definition makes the logic trivializable because $\neg\, a \approx a \equiv \bot$

The Equality Relation $\approx$

- **Joining non-trivial sets may result in a trivial set**
  - $\Sigma_1 = \{(\leq 3R.C)\}$
  - $\Sigma_2 = \{(\geq 5R.C)\}$
  - $\Sigma_3 = \Sigma_1 \cup \Sigma_2$
- **Difficult to check a set of formulas for expressions that may trivialize the set**

Proposal for Redefinition of Equality – Open for Discussion

- **Alternative definition for semantics of $\approx$**
  - $I, \sigma \models_t x \approx y$ **iff** $x^{I,\sigma} = y^{I,\sigma}$
  - $I, \sigma \models_f x \approx y$ **for all x and y**
- $\{a \approx b, \neg\, a \approx b\}$ **is not a trivial set**
  - $\{a \approx b, \neg\, a \approx b\} \models_t \neg\, a \approx a$ **and**
  - $\{a \approx b, \neg\, a \approx b\} \models_t a \approx a$
- **Negation of $a \approx b$ cannot be expressed by $\neg$ but as *failed to prove* $a \approx b$**

Summary and Outlook

- **Paraconsistent first-order logic successfully designed**
- **Paraconsistent $\mathcal{ALC}$ successfully designed**
- **Paraconsistent DL reasoning for KAON2 implemented**
- **Paraconsistent $\mathcal{SHIQ}$ in discussion**
  - **Equality is required but problematic**

Proposals for Discussion

- **Definition and use of the equality relation $\approx$**
- **Equivalence connective $\equiv$ in description logics**
- **Mapping to classical logic**
- **Other proposals for application to DLs**

References

- **Ofer Arieli and Arnon Avron,**
  - ***The Value of the Four Values***
- **Peter F. Patel-Schneider,**
  - ***A Four-Valued Semantics for Terminological Logics***
- **Umberto Straccia,**
  - ***A Sequent Calculus for Reasoning in Four-valued Description Logics***

Definition of Ontology

An ontology defines the terms used to describe and represent an area of knowledge.

Ontologies are used to share domain information.

Ontologies include computer-usable definitions of basic concepts in the domain and the relationships among them.

We abstract the term ontology as a set of formulas of a logic.

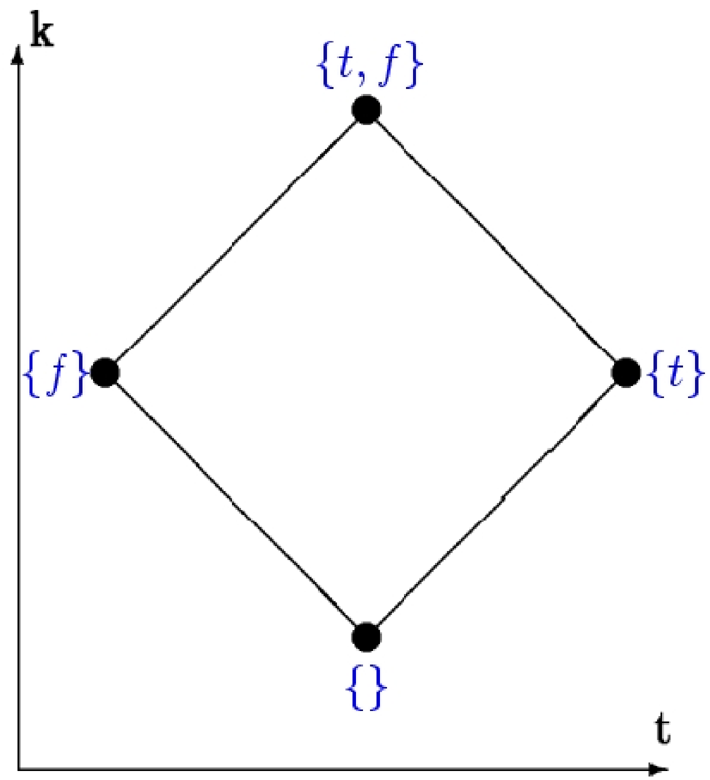
Model and Interpretation

Let $\langle \mathcal{L}, \models \rangle$ be a logic, $\mathcal{L}$ its set of formulas, $\models$ its entailment relation and let Γ be a set of formulas

A *model* of Γ is an interpretation that evaluates all $F \in \Gamma$ to be true

An *interpretation* I consists of a non-empty domain $\mathcal{D}$ and a mapping ^{I} that assigns truth values to formulas of $\mathcal{L}$

Representation by a Logical Bilattice



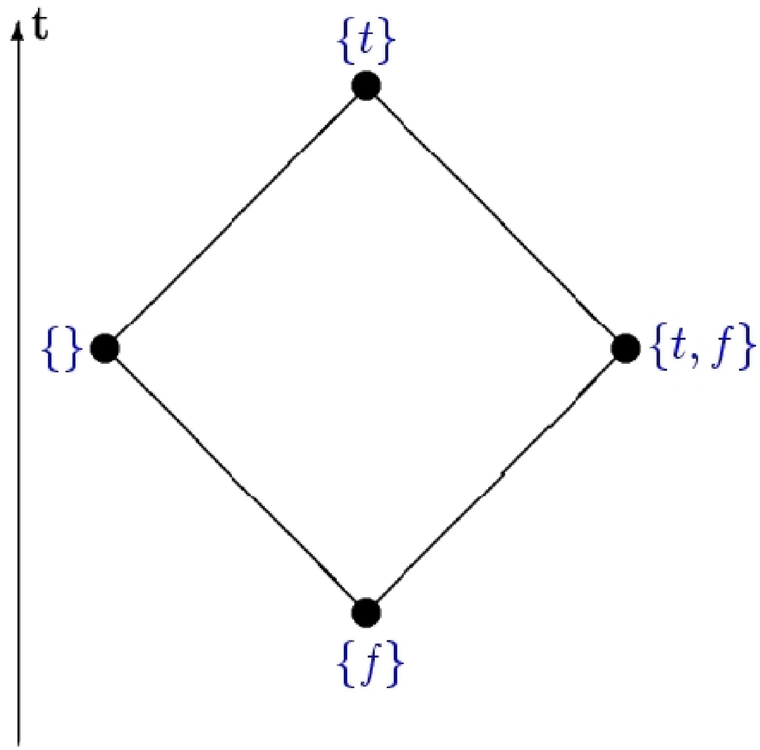
meet-operation on truth-lattice corresponds to *conjunction*

join-operation on truth-lattice corresponds to *disjunction*

***Negation* is defined as:**

α	$\{\}$	$\{f\}$	$\{t\}$	$\{t, f\}$
$\neg\alpha$	$\{\}$	$\{t\}$	$\{f\}$	$\{t, f\}$

The Truth-Lattice



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α	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\neg\alpha$	$\{\}$	$\{t\}$	$\{f\}$	$\{t,f\}$

Representation of a simple Equivalence

The four-valued first-order logic formula

$$F_4 = R(x) \Leftrightarrow S(x)$$

Representation in classical first-order logic by

$$\begin{aligned} F_2 \quad = \quad & ((\neg R_-(x) \wedge \neg S_+(x)) \vee R_+(x)_+ \vee S_-(x)) \\ & \wedge((\neg R_+(x) \wedge \neg S_-(x)) \vee R_-(x) \vee S_+(x)) \\ & \wedge(R_+(x) \leftrightarrow S_+(x)) \wedge (R_-(x) \leftrightarrow S_-(x)) \end{aligned}$$

Three-Valued Implication $\rightarrow_3$

- Definition of $\rightarrow_3$ with the properties
  - Full paraconsistency
  - Deduction theorem holds
  - Possible negation normal form transformation

$\rightarrow_3$	$\{f\}$	$\{t\}$	$\{t, f\}$
$\{f\}$	$\{t, f\}$	$\{t\}$	$\{t, f\}$
$\{t\}$	$\{f\}$	$\{t\}$	$\{t, f\}$
$\{t, f\}$	$\{f\}$	$\{t\}$	$\{t, f\}$

Negation Normal Form Representation of $\longrightarrow_3$

The three-valued first-order logic formula

$$R(x) \longrightarrow_3 S(x)$$

Additional involution connective $\sim_3$

α	$\{f\}$	$\{t\}$	$\{t, f\}$
$\sim_3 \alpha$	$\{t, f\}$	$\{t\}$	$\{f\}$

Transformation in negation normal form

$$R(x) \longrightarrow_3 S(x) \quad \rightsquigarrow \quad R(x) \vee \neg S(x) \wedge \sim_3 \neg S(x)$$

The Deduction Theorem

Let $\models$ be an entailment relation, let $\rightarrow$ be an implication connective,

Let Γ be a set of formulas, and let F and G be formulas

Deduction Theorem states relationship between $\models$ and $\rightarrow$

$$\Gamma \cup \{F\} \models G \quad \textbf{iff} \quad \Gamma \models F \rightarrow G$$

Truth Table Connectives Satisfying Deduction Theorem

$\varobslash$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\{\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{t\}$	$\{\}$ / $\{f\}$	$\{\}$ / $\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{t,f\}$	$\{\}$ / $\{f\}$	$\{\}$ / $\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$

Truth Table Connectives Satisfying Deduction Theorem

$\odot$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\{\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{t\}$	$\{\}$ / $\{f\}$	$\{\}$ / $\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{t,f\}$	$\{\}$ / $\{f\}$	$\{\}$ / $\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$

$\odot$ **is a non-monotonic connective**

Truth Table Connectives Satisfying Deduction Theorem

$\ogreaterthan$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\{\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{t\}$	$\{\}$ / $\{f\}$	$\{\}$ / $\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$
$\{t,f\}$	$\{\}$ / $\{f\}$	$\{\}$ / $\{f\}$	$\{t\}$ / $\{t,f\}$	$\{t\}$ / $\{t,f\}$

$\rightarrow$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$
$\{\}$	$\{t\}$	$\{t\}$	$\{t\}$	$\{t\}$
$\{f\}$	$\{t\}$	$\{t,f\}$	$\{t\}$	$\{t,f\}$
$\{t\}$	$\{\}$	$\{\}$	$\{t\}$	$\{t\}$
$\{t,f\}$	$\{\}$	$\{f\}$	$\{t\}$	$\{t,f\}$

Definition of Monotonicity

Let $\langle \mathcal{D}, \leq \rangle$ be a partially ordered set.

An operation $f : \mathcal{D}^n \to \mathcal{D}$ is called *monotonic*

(with respect to $\leq$) if $f(x) \leq f(y)$ whenever $x \leq y$

(we say that $x = (x_1, ..., x_n) \leq y = (y_1, ..., y_n)$

iff we find for all $1 \leq i \leq n$ that $x_i \leq y_i$).

We define monotonicity w.r.t. to knowledge lattice $\leq_k$