BCU Mathematics Contest 2000
Problems

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Contest Web Page:
http://maths.ucc.ie/~pascal/verein/internat/index.html

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INDIVIDUAL CONTEST PAPER

Answer *all* 4 questions.

Contestants have 45 minutes to do the work.

1 Solve in real numbers:

$$\frac{|3x-2|}{|x|} < 2 - \frac{1}{|x|}$$

[15 points]

2 Let f be the function $f : x \mapsto \sin 2x$ defined on the real numbers. For each positive integer n, let $f^{(n)}$ denote the function which is the n-th derivative of f. Determine all intersections with the x-axis, local maxima, local minima, and inflection points of f and all $f^{(n)}$.

[20 points]

3 Let F be a square with vertices $ABCD$ and P a pyramid erected on F with top apex E. If M_1 is the intersection of the diagonals in F, M_2 is the midpoint of CD and S is the centroid of the triangle ABE, prove that M_1E and M_2S intersect at a point T.

[7 points]

4 Let f be a function which maps pairs of real numbers to real numbers, i.e. if x, y are real numbers, then $f(x, y)$ is a real number.

Assume that f satisfies the following two conditions:

(1) For each real number x we have $f(x, x) = 0$.
(2) For all real numbers x, y, z we have $f(x, z) \leq f(x, y) + f(z, y)$.

Show that f satisfies the following two properties:

(a) For all real numbers x, y we have $f(x, y) \geq 0$.
(b) For all real numbers x, z we have $f(x, z) = f(z, x)$.

Give an example of such a function.

[12 points]

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TEAM CONTEST PAPER

Answer *all* 3 questions.

Contestants have 45 minutes to do the work.

Teams are permitted to submit *at most* one solution per question. It's therefore up to each team to select a solution that, in their opinion, will attract the maximum score.

1 Three circles of equal radius are centered on the vertices of an equilateral triangle. The diameters of the circles are such that the three circles intersect at a single point inside the triangle. What is the total area of the "shamrock"-shaped figure enclosed by the three circles if each side of the equilateral triangle has length a?

[14 points]

2 Determine the area of the bounded region enclosed by the following three curves in the real plane:

$$
\begin{aligned}
y &= \sqrt{x + 4\pi^2}, \\
y &= -\sqrt{x + 4\pi^2} \text{ and} \\
\sin y &= x.
\end{aligned}
$$

[16 points]

3 For a contest, n students are supposed to form groups of 3 or 5 members (groups of 4 are not allowed).

(a) For which n is this possible, such that each student is in exactly one group?

(b) Determine the number of groups if as many as possible of them have 3 members. What then is the average size of a group?

[15 points]

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SPEED CONTEST PAPER FOR TEAMS

Contestants have 45 minutes to attempt a maximum of 7 questions.
Questions will be distributed one at a time.

Only the team which hands up the correct solution in the shortest time earns the marks assigned to a particular question; all other teams earn no marks on that question. Each incorrect attempted answer will result in the *reduction* of 1 point. As soon as a correct answer to a question is submitted to the invigilators the next question will be distributed to all the teams. The team score for this contest will be the sum of its scores on the questions handed out in the time period set for the contest.

1 How many different polynomials arise from $p_0(x) = x^2 + 1,\ p_1(x) = x^2$ and

$$p_{n+1}(x) = -p_n(x) - p_{n-1}(x),\ n = 1, 2, \ldots$$

List all polynomials.

[3 points]

2 If the number 1503985847 is divided by $934152a$ (where a stands for the last digit of the number), a positive integer is obtained. Determine all possible values of a.

[3 points]

3 A function f is defined on the positive integers as follows: If $n > 100$, then $f(n) = n - 10$. If $n \leq 100$, then $f(n)$ is the value of $f(y)$ where $y = f(n + 11)$. Determine $f(98)$.

[5 points]

4 Three players A, B and C play a game according to the following rules.

In the first round, player A selects a number a, then player B selects a number b and finally player C selects a number c.

In the second round, the players A, B and C again select in turn three numbers a', b' and c'. In each round, player B knows which number A has selected and player C knows which numbers A and B have selected.

The player B wins if the resulting system of equations

$$\begin{aligned} ax + y + bz &= c \\ a'y + b'z &= c' \end{aligned}$$

has at least one solution.

Determine a choice for B which insures that he wins.

[5 points]

5 An isosceles triangle $\triangle ABC$ with base AB and equal sides AC, BC has height 5cm and the length of AB is 4cm. Points D on AC and E on BC are chosen such that DE is parallel to AB, and such that the rectangle inscribed in the trapezoid $ABED$, with one side being DE and the other on AB, has maximal area. Determine the length of DE.

[7 points]

6 Determine all triples of real numbers $x, y, z \in [0, \pi]$ such that

$$\sin x \sin y \sin z = 1.$$

[3 points]

7 Suppose α, β, γ are the roots of the cubic polynomial $p(x) = x^3 - x - 1$. Determine the product of the three numbers

$$\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}, \frac{1-\gamma}{1+\gamma}.$$

[5 points]