BCU Mathematics Contest 1999

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1. Since a non-leap year has 365 days, we’re required to find positive integers $a, b$ such that

$$ab + a + b = 365.$$  

Now $ab + a + b = (a + 1)(b + 1) - 1$, hence $a, b$ are positive integral solutions of the equation

$$(a + 1)(b + 1) = 366 = 1.2.3.61.$$  

Since $a, b$ are positive, and the equation is symmetric in $a, b$ the only possible solution pairs are $(1, 182), (2, 121), (5, 60)$. The answer is therefore 3.

Contestants who find these solutions by trial and error, and do not show that these are the only solutions, get a maximum of 4 points.

2. If

$$a_n = \frac{A}{n - 1} + \frac{B}{n + 1}, \quad n = 2, 3, \ldots,$$

then, in particular,

$$a_2 = A + \frac{1}{3}B, \quad a_3 = \frac{1}{2}A + \frac{1}{4}B,$$

i.e.,

$$\frac{2}{3} = A + \frac{1}{3}B, \quad \frac{1}{4} = \frac{1}{2}A + \frac{1}{4}B,$$

whence

$$2 = 3A + B, \quad 1 = 2A + B.$$
Thus $A = 1$ and $B = -1$. It remains to show that these values work for all $n \geq 2$. But this is easy: if $n \geq 2$, then

\[
\frac{1}{n-1} + \frac{-1}{n+1} = \frac{n+1-(n-1)}{(n-1)(n+1)} = \frac{2}{n^2-1} = a_n.
\]

OR

\[
\frac{2}{n^2-1} = \frac{2}{(n-1)(n+1)} = \frac{A}{n-1} + \frac{B}{n+1} = \frac{A(n+1) + B(n-1)}{(n-1)(n+1)} = \frac{(A + B)n + (A - B)}{n^2 - 1},
\]

for $n \geq 2$ iff $2 = (A + B)n + A - B$, $n = 2, 3, \ldots$. This is so iff $A = -B = 1$.

We now have for $k = 2, 3, \ldots$

\[
s_k = \sum_{n=2}^{k} a_n = \sum_{n=2}^{k} \left[ \frac{1}{n-1} - \frac{1}{n+1} \right] = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{k-2} - \frac{1}{k}\right) + \left(\frac{1}{k-1} - \frac{1}{k+1}\right).
\]

Removing brackets and cancelling we see that

\[
s_k = 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} = \frac{3}{2} - \frac{2k+1}{k(k+1)}.
\]
Clearly, then
\[
s_{100} - s_{10} = \frac{21}{10.11} - \frac{201}{100.101} < \frac{21}{10.11} < \frac{1}{5}\]

and
\[
s_{100} - s_{10} = \frac{1}{10} + \frac{1}{11} - \left( \frac{1}{100} + \frac{1}{101} \right) < \frac{2}{10} = \frac{1}{5}\]

Thus the error lies between \(4/25\) and \(1/5\).
3. 

\[ y' = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x - 2)^2 \geq 0, \forall x. \]

Hence \( y \) is increasing on \((-\infty, \infty)\). Also,

\[ y'' = 6x - 12x = 6(x - 2). \]

Clearly, \( y'' \geq 0, \forall x \geq 2 \) and \( y'' \leq 0, \forall x \leq 2 \). Hence \( y \) is convex if \( x \geq 2 \) and concave if \( x \leq 2 \). It has no local extrema and one point of inflection at \( x = 2 \). Also,

\[ \lim_{x \to \infty} = \infty, \quad \lim_{x \to -\infty} = -\infty. \]

Since \( y \) is a cubic polynomial with real coefficients it crosses the \( x \)-axis at least once. To see where, note that

\[ y = x^3 - 6x^2 + 12x - 7 = (x - 2)^3 + 1 = 0 \]

if \( x = 1 \). (Alternatively, the root can be found by trial and error.) Then

\[ y = (x - 1)(x^2 - 5x + 7) = (x - 1)((x - 5/2)^2 + 3/4). \]

Hence \( y \) crosses the horizontal axis precisely once.

OR

It is \( x^3 - 6x^2 + 12x - 7 = x^3 - 3 \cdot 2x^2 + 3 \cdot 2^2 x - 2^3 + 1 = (x - 2)^3 + 1 \). Therefore, the function has the equation

\[ y = (x - 2)^3 + 1, \]

i.e. it is the cubic function \( y = x^3 \) where the point of inflection is moved to \((2, 1)\). Hence, it is convex if \( x \geq 2 \), concave if \( x \leq 2 \), and has point of inflection \((2, 1)\).

4. Determine the area of the bounded region enclosed by the set of points in the \( xy \)-plane that satisfy the equation

\[ xy + y^2 - yx^2 - x^3 = 0. \]
Note that \( xy + y^2 - yx^2 - x^3 = (y + x)(y - x^2) \). Thus \((x, y)\) belongs to the set

\[ \{(x, y) : xy + y^2 - yx^2 - x^3 = 0 \} \]

iff it lies on the line \( y = -x \) or on the parabola \( y = x^2 \). The bounded region enclosed by these curves is therefore the set

\[ \{(x, y) : x^2 \leq y \leq -x, -1 \leq x \leq 0 \}, \]

the area of which is

\[ \int_{-1}^{0} (-x) \, dx - \int_{-1}^{0} x^2 \, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \]

OR

Not everybody will spot the factorization. Some may treat \( xy + y^2 - yx^2 - x^3 = 0 \) as a quadratic in \( y - y^2 + (x - x^2)y - x^3 = 0 \)—and solve for \( y \):

\[
\begin{align*}
y &= \frac{-(x - x^2) \pm \sqrt{(x - x^2)^2 + 4x^3}}{2} \\
&= \frac{-(x - x^2) \pm \sqrt{x^2 + 2x^3 + x^4}}{2} \\
&= \frac{-(x - x^2) \pm |x + x^2|}{2}.
\end{align*}
\]
Now denoting the roots by $y_+, y_-$ we see that

$$y_+ = \begin{cases} x^2, & \text{if } x(1 + x) \geq 0 \\ -x, & \text{if } x(1 + x) \leq 0, \end{cases}$$

and

$$y_- = \begin{cases} -x, & \text{if } x(1 + x) \geq 0 \\ x^2, & \text{if } x(1 + x) \leq 0, \end{cases}$$

It follows that the set

$$\{(x, y) : xy + y^2 - yx^2 - x^3 = 0\}$$

is the union of the line $y = -x$ and the parabola $y = x^2$. The problem can be finished as before.

Inevitably, some of those who treat it this way will say that $\sqrt{(x + x^2)^2} = x + x^2$ and simplify the work for themselves! But they should be penalised!
Figure 2: The two graphs enclosing the area in question.
Volume:

Any of the pyramids which is cut away at some edge has volume $V_P = \frac{a^3}{6}$. The volume of the remaining solid is therefore

$$V_R = 8a^3 - \frac{4}{3}a^3 = \frac{20}{3}a^3.$$

Surface:

Along the cuts appear equilateral triangles with sides of length $a\sqrt{2}$. Hence, each of those triangles has area $A_T = \frac{\sqrt{2}}{2}\sqrt{3}$. The surface area
of the remaining solid is therefore

\[ O_R = 24a^2 - 8 \cdot 3 \cdot \frac{a^2}{2} + 8 \cdot \frac{a^2}{2} \sqrt{3} = 12a^2 + 4a^2 \sqrt{3}. \]

2. The largest root of the equation is

\[ \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \]

For \( a \to 0 \), both numerator and denominator tend to 0.

First solution:

\[ \beta = \frac{(\sqrt{b^2 - 4ac} - b)(\sqrt{b^2 - 4ac} + b)}{2a(\sqrt{b^2 - 4ac} + b)} = \frac{-2c}{\sqrt{b^2 - 4ac} + b} \]

and therefore

\[ \lim_{a \to 0} \beta = -\frac{c}{b}. \]

Second solution:

Using de l’Hospital’s rule we obtain

\[ \lim_{a \to 0} \beta = \lim_{a \to 0} \frac{-2a}{2(\sqrt{b^2 - 4ac})} = \frac{-c}{b}. \]

Third Solution:

Let \( h = -4ac \) and, for \( x > 0 \), define \( f(x) = \sqrt{x} \). Then \( b = f(b^2) \) and

\[ \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -2c \left( \frac{f(b^2 + h) - f(b^2)}{h} \right). \]

Now as \( a \to 0 \), \( h \to 0 \) and so

\[ \lim_{a \to 0} \beta = -2c \lim_{h \to 0} \frac{f(b^2 + h) - f(b^2)}{h} \]

\[ = -2cf'(b^2) \]

\[ = -2c \cdot \frac{1}{2\sqrt{b^2}} \]

\[ = \frac{c}{b}. \]
since
\[ f'(x) = \frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}, \quad \forall x > 0. \]

Comment. All three proofs rely on the crucial fact that \( f \) is continuous on \([0, \infty)\).

3. First Solution. Join \( B \) to \( E \) and to \( Y \). Let \( \theta = \angle YAX \) and \( \phi = \angle ACX \), so that \( \phi = \pi/2 - \theta \). Then since \( \angle BXA = \angle BYA = \pi/2 \) we see that \( \angle CBY = \theta \). But the triangles \( BCY \) and \( BYE \) are similar. Hence \( \angle YBE = \angle CBY = \theta \) and \( \angle BEC = \phi \). It now follows that

\[ \angle ABE = \angle ABC + \angle CBY + \angle YBE = \phi - 2\theta + \theta + \theta = \phi. \]

Hence the triangle \( AEB \) is isosceles and so \( |AE| = |AB| = d \).

![Diagram](https://via.placeholder.com/150)

**Figure 4:** Picture for Question 3.

Second Solution. Let \( x = |AC|, y = |CY| \). Then

\[ x \cos \theta = |AX| = d \cos \angle BAX = d \cos 3\theta, \]

and

\[ x + y = |AY| = d \cos \angle BAY = d \cos 2\theta. \]
Hence

\[ y = d[\cos 2\theta - \cos 3\theta] = \frac{d(\cos 2\theta \cos \theta - \cos 3\theta)}{\cos \theta}, \]

and so

\[ |AE| = x + 2y = \frac{d(2 \cos 2\theta \cos \theta - \cos 3\theta) \cos \theta}{\cos \theta} = \frac{d(2 \cos 2\theta \cos \theta - \cos 2\theta \cos \theta + \sin 2\theta \sin \theta) \cos \theta}{\cos \theta} = \frac{d(\cos 2\theta \cos \theta + \sin 2\theta \sin \theta) \cos \theta}{\cos \theta} = \frac{d \cos \theta}{\cos \theta} = \frac{d}{2}. \]
BCU MATHEMATICS CONTEST, MARCH 13, 1999

SOLUTIONS FOR THE SPEED CONTEST PAPER
FOR TEAMS

1. Let \( a \) be the number of initial participants. Then, initially, we have

\[
\frac{55}{100} \cdot a
\]
girls among them. We therefore obtain the equation

\[
\frac{60}{100} (a + 5) = \frac{55}{100} a + 5
\]

and therefore

\[
a = 40.
\]

So, initially, there were 40 participants and \( \frac{55 \cdot 40}{100} = 22 \) girls among them.

2. For all pairs \((x, y)\) with \(x^2 + y^2 = 1\) we obtain

\[
f(x, y) = 2(x^2 + y^2) + y^2 = 2 + y^2.
\]

Since the condition on \( x \) and \( y \) implies that \( 0 \leq y^2 \leq 1 \), we have

\[
2 \leq f(x, y) \leq 3.
\]

We obtain the maximum value 3 for \( y = -1 \) and \( y = 1 \), and the minimum value 2 for \( y = 0 \). The solutions therefore are:

Maxima: \((0, -1), (0, 1)\)

Minima: \((1, 0), (-1, 0)\)
3. Since $1 \leq b < a$, we have $b \leq b^2 < a^2$ and so

$$b - a^2 < b - b^2 < b < a < 2a$$

which suffices.

OR

$$b - a^2 < b - b^2 \leq b - b = 0 < 1 \leq a < 2a.$$ 

4. The height $h$ of the triangle is

$$h = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{a}{2} \sqrt{3},$$

\begin{figure}[h]
    \centering
    \includegraphics[width=0.5\textwidth]{figure5.png}
    \caption{Question 4}
\end{figure}

Since the altitudes of the triangle intersect at a third of their lengths, and the centres of the circles coincide at the point of intersection of the altitudes, we get

$$r_i = \frac{a}{6} \sqrt{3}$$

as the radius of the inscribed circle, and

$$r_c = \frac{a}{3} \sqrt{3}$$
as the radius of the circumscribed circle.

5. For \( x = 0 \) the recursion formula reduces to
\[
p_{n+2}(0) = p_{n+1}(0).
\]
Hence we obtain
\[
\begin{align*}
p_1(0) &= 1, \\
p_2(0) &= 3, \\
p_3(0) &= 3, \\
p_5(0) &= p_4(0) = p_3(0) = 3.
\end{align*}
\]

For \( x = -1 \) the recursion formula is
\[
p_{n+2}(-1) = p_{n+1}(-1) - 2p_n(-1),
\]
and therefore
\[
\begin{align*}
p_1(-1) &= 1, \\
p_2(-1) &= 2, \\
p_3(-1) &= 0, \\
p_4(-1) &= -4, \\
p_5(-1) &= -4.
\end{align*}
\]

6. We obtain
\[
\begin{align*}
7^1 &= \ldots 07 \\
7^2 &= \ldots 49 \\
7^3 &= \ldots 43 \\
7^4 &= \ldots 01 \\
7^5 &= \ldots 07 \\
7^6 &= \ldots 49 \\
&\vdots \\
7^{76} &= \ldots 01
\end{align*}
\]
since 76 mod 4 = 0. Hence, the last two digits are 01.
7. Each of the 27 students taking three courses takes a course in Mathematics or Computer Science. Since 34 students take Spanish or History, \(45 - 34 = 11\) students take Mathematics or Computer Science, but neither of Spanish or History. Each of the latter group of students therefore attends a maximum of two courses. From this we obtain a lower bound of \(27 + 11 = 38\) students taking Mathematics or Computer Science.

We show that the bound is sharp, by giving a distribution of students among the courses such that exactly 38 students take Mathematics or Computer Science. It is as follows:

- 27 students take 3 courses.
- 11 students take Mathematics and Computer Science only.
- 7 students take Spanish and History only.

With this distribution, the assumptions of the question are satisfied and 38 students take Mathematics or Computer Science. Hence, the bound is sharp.