CS 740 – Computational Complexity and Algorithm Analysis

Spring Quarter 2010

Slides 2

Pascal Hitzler
Kno.e.sis Center
Wright State University, Dayton, OH
http://www.knoesis.org/pascal/

Contents

1. SAT is in NP
2. SAT is NP-hard

SAT is in NP

$$F = \left(\bigwedge_{i=1}^{n} \left(\bigvee_{j=1}^{m} L_{i,j} \right) \right)$$

- **Non-deterministically pick a truth assignment.**
  Represent this in a look-up table.
  [linear in number of literals]
- **Check if truth assignment satisfies F.**
  [quadratic – because of comparison of input with table entries]

- **Formally, we need to do this on a TM – the encoding is a bit unwieldy, but straightforward.**

Contents

1. SAT is in NP
2. SAT is NP-hard

Idea

- Give a logical formula which transforms computations of a TM M with input string u into a formula f(u) s.t.

  u is accepted iff f(u) is satisfiable.

- + show that transformation is polynomial.

- [f(u) doesn’t have to be in CNF because of Exercise 30]

Encoding

ND TM M:

- states: $q_0,\ldots,q_m$
- alphabet: $B=a_0,\ldots,a_t$
- accepting state: q_m
- rejecting state: q_{m-1} (only one)

p(n) polynomial which is upper bound to number of computations

Boolean variables:

- $Q_{i,k}$ M is in state q_i at time k
- $P_{j,k}$ Tape head is in position j at time k
- $S_{j,r,k}$ Tape position j contains symbol a_r at time k

SAT is NP-hard: Clauses *i* & *ii*

	Clause	Conditions	Interpretation
i)	State $\bigvee_{i=0}^{m} Q_{i,k}$	$0 \le k \le p(n)$	For each time k, M is in at least one state [p(n) clauses, m literals each]
	$\neg Q_{i,k} \vee \neg Q_{i',k}$	$0 \le i < i' \le m$ $0 \le k \le p(n)$	M is in at most one state at any time [O(m²) × p(n) clauses]
ii)	Tape head $\bigvee_{j=0}^{p(n)} P_{j,k}$	$0 \le k \le p(n)$	For each time k, the tape head is in at least one position [p(n) clauses, p(n) literals each]
	$\neg P_{j,k} \vee \neg P_{j',k}$	$0 \le j < j' \le p(n)$ $0 \le k \le p(n)$	…and at most one position [O(p(n)³) clauses]

SAT is NP-hard: Clause *iii*

	Clause	Conditions	Interpretation
iii)	Symbols $\bigvee_{r=0}^{t} S_{j,r,k}$	$0 \leq j \leq p(n)$ $0 \leq k \leq p(n)$	For each time k and position j, position j contains at least one symbol [$p(n)^2$ clauses, t literals each]
	$\neg S_{j,r.k} \vee \neg S_{j,r',k}$	$0 \leq j \leq p(n)$ $0 \leq r < r' \leq t$ $0 \leq k \leq p(n)$	… and at most one symbol [$O(t^2) \times p(n)^2$ clauses]

SAT is NP-hard: Clauses *iv* & *v*

	Clause	Interpretation
iv)	Initialization $Q_{0,0}$	Begin in state 0
	$P_{0,0}$	…reading leftmost tape cell (position 0)
	$S_{0,0,0}$	…which contains a blank (symbol 0)
	$S_{1,r1,0}$	The next n symbols contain the input string,
	$S_{2,r2,0}$	which we'll denote $a_{r1}, a_{r2}, \ldots a_{rn}$
	…	
	$S_{n,rn,0}$	
	$S_{n+1,0,0}$	And the rest of the tape contains blanks…
	…	
	$S_{p(n),0,0}$	… for the entire accessible portion
v)	Final state $Q_{m,p(n)}$	The computation ends in q_m – the accepting state

A computation that satisfies all of these clauses still doesn't necessarily follow the rules of the machine, M.

Each state/symbol/position after time 0 must be obtained from the transition rules of M.

Tape Consistency

	Clause	Conditions	Interpretation
vi)	Tape Changes $\neg S_{j,r,k} \vee P_{j,k} \vee S_{j,r,k+1}$	$0 \leq j \leq p(n)$ $0 \leq r \leq t$ $0 \leq k \leq p(n)$	Symbols not at the position of the tape head are unchanged [$p(n)^2 \times t$ clauses]

Converting rules in δ to clauses

$$\underbrace{\neg Q_{i,k} \vee \neg P_{j,k} \vee \neg S_{j,r,k}} \vee Q_{i',k+1}$$

If none of these are satisfied, then we are in state q_i and position j scanning symbol a_r at time k

In that case, the next state must be $Q_{i'}$ or the clause is not satisfied.

For <u>each</u> $\delta(q_i, a_r) = [q_{i'}, ?, ?]$

Same thing for tape symbols

$$\neg Q_{i,k} \vee \neg P_{j,k} \vee \neg S_{j,r,k} \vee S_{j,r',k+1}$$

If none of these are satisfied, then we are in state Q_i and position P_j scanning symbol S_r at time k

In that case, the next symbol at position j must be $S_{r'}$ or the clause is not satisfied.

For each $\delta(q_i, a_r) = [?, a_{r'}, ?]$

Same thing for tape head position

$$\neg Q_{i,k} \vee \neg P_{j,k} \vee \neg S_{j,r,k} \vee P_{j+n(d),k+1}$$

If none of these are satisfied, then we are in state Q_i and position P_j scanning symbol S_r at time k

In that case, the tape head will move either one position left or one position right.

Where n(L) = -1, and n(R) = +1
For each $\delta(q_i, a_r)$ = [?, ?, L/R]

Consistency

The conjunction of these three clause types ensures that if we are in a certain state, reading a particular symbol at a particular time, we must be in the right configuration, according to δ in the following time step.

These are <u>machine dependent.</u>

Consistency clauses

Consistency clauses are constructed for every time, state, tape head position and tape symbol.

However, if we are scanning position 0 and attempt to move left, we go directly to the rejecting state.

Hey, wait a minute!

We've been talking like there is only one transition for each state/symbol pair, but this is a <u>non-deterministic</u> Turing machine, right?

Let trans(i, j, r, k) be the disjunction of all the consistency clause sets for i, j, r, k. The resulting clause ensures that we are in <u>some</u> valid configuration following each transition.

And now we're done

Clause	Interpretation
vi) <u>Halted</u>	
$\neg Q_{i,k} \vee \neg P_{j,k} \vee \neg S_{j,r,k} \vee Q_{i,k+1}$	same state
$\neg Q_{i,k} \vee \neg P_{j,k} \vee \neg S_{j,r,k} \vee P_{j,k+1}$	same tape head position
$\neg Q_{i,k} \vee \neg P_{j,k} \vee \neg S_{j,r,k} \vee S_{j,r,k+1}$	same symbol at position r

For all appropriate j, r, k, and i = q_{m-1},
and i = q_m

What we've done so far…

We've defined a set of wff that are satisfiable if (and only if) some computation of ND TM M leads to an accepting final state.

Polynomial transformation?

Can the formula be created from any NDTM M *in polynomial time?*

- **The values *m* and *t* are independent of the size of the input string. They don't grow with *n*.**
- **The number of clauses is polynomial in *p*(*n*).**

qed