Characterizing $\Phi$-accessible Programs

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Abstract

We characterize programs with a total Fitting semantics, recovering from some mistakes in [Hit01].
Notation and terminology is that of [Hit01].

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Roland Heinze found the problem with the original definition, see Remark 0.4.

0.1 Theorem Let $P$ be a normal logic program. The following conditions are equivalent.
1) $P$ has a total Fitting semantics.
2) There exists a model $I$ and a level mapping $l$ such that $I$ is a supported model of $P$ and each $A \in B_P$ satisfies either (i) or (ii).
   (i) There exists a clause $A \leftarrow L_1, ..., L_n$ in ground($P$) with head $A$ such that $I \models L_1 \land \cdots \land L_n$, and $l(A) > l(L_i)$ for all $i$.
   (ii) For each clause $A \leftarrow L_1, ..., L_n$ in ground($P$) with head $A$ there exists $i$ such that $I \not\models L_i$ and $l(A) > l(L_i)$.
3) There exists a model $I$ and a level mapping $l$ such that $I$ is a model of $P$ and each $A \in B_P$ satisfies either (i) from 2) above or (iii).
   (iii) $I \not\models A$ and for each clause $A \leftarrow L_1, ..., L_n$ in ground($P$) with head $A$ there exists $i$ such that $I \not\models L_i$ and $l(A) > l(L_i)$.

Furthermore, $I$ is the unique supported model of $P$.  

**Proof:** 3) ⇒ 2). Suppose 3) holds. We show that $I$ is supported. So let $A \in I$. Then (iii) cannot hold. So (i) holds by condition 3). Hence there exists a clause in ground($P$) with head $A$ whose body is true under $I$. So $I$ is supported. Furthermore, if $A \in I$, then (i) holds. If $A \notin I$, then (i) can not hold since $I$ is a model for $P$. Hence (iii) holds by condition 3). Consequently, (ii) holds in this case.

2) ⇒ 3). Suppose $A$ does not satisfy (i). Then it satisfies (ii). Since $I$ is supported, we must have $A \notin I$ which shows (iii).

1) ⇔ 2). This is exactly the proof of Theorem 6.5.3 in [Hit01]. We replicate it here.

1) ⇒ 2). Suppose $P$ satisfies condition 1). For each $A \in B_P$, let $l_p(A)$ denote the least ordinal $\beta$ such that $A$ is not undefined in $\Phi_P \uparrow (\beta + 1)$. Let $a$ be its closure ordinal wrt. $\Phi_P$ and let $M_P = \Phi_P \uparrow a^+$ be its unique supported (two-valued) model. We distinguish two cases (a) and (b).

(a) Let $A \in M_P$ and $l_p(A) = \beta$. By definition of $l_p$ and $\Phi_P$ there exists a clause $A \leftarrow L_1, \ldots, L_n$ in ground($P$) such that the $L_1, \ldots, L_n$ are true in $\Phi \uparrow \beta$, and hence are also true in $M_P$. Again by definition of $l_p$ we obtain $l_p(A) > l_p(L_i)$ for all $i$.

(b) Let $A \notin M_P$ and $l_p(A) = \beta$. By definition of $l_p$ and $\Phi_P$ we obtain that for any clause $A \leftarrow L_1, \ldots, L_n$ in ground($P$) we must have that $L_1 \land \cdots \land L_n$ is false in $\Phi_P \uparrow \beta$. So there must be some $i$ such that $L_i$ is false in $\Phi_P \uparrow \beta$ and $l(L_i) < \beta$ by definition of $l_p$, and hence $l_p(A) > l_p(L_i)$.

Thus, $P$ satisfies condition 2) with $I = M_P$ and $l = l_p$.

2) ⇒ 1) Assume $P$ satisfies condition 2). We show by induction on $\beta$ that any $A \in B_P$ with $l(A) = \beta$ is not undefined in $\Phi_P \uparrow (\beta + 1)$ and, furthermore, that $I$ and $\Phi_P \uparrow (\beta + 1)$ agree on $A$.

If $l(A) = 0$, then $A$ must be the head of a unit clause or does not appear in any head. In the first case, $A$ is true in $\Phi_P \uparrow 1$, and in the second case, $A$ is false in $\Phi_P \uparrow 1$. Note that in the first case $A$ is also true in $I$ since condition (i) applies and $I$ is a model of $P$. Also, in the second case, $A$ is also false in $I$ since condition (ii) applies and $I$ is supported.

Now let $l(A) = \beta$. If there is no clause in ground($P$) with head $A$, then $A$ is false in $\Phi_P \uparrow 1 \leq \Phi_P \uparrow (\beta + 1)$ and also false in $I$ since condition (ii) applies and $I$ is supported. So assume there is a clause in ground($P$) with head $A$. By hypothesis, either condition (i) or condition (ii) applies.

If condition (i) applies, then there is a clause $A \leftarrow L_1, \ldots, L_n$ in ground($P$) such that $l(L_1), \ldots, l(L_n) < l(A)$ and therefore, by the induction hypothesis, the $L_1, \ldots, L_n$ are not undefined in $\Phi_P \uparrow \beta$ and $I$ agrees with $\Phi_P \uparrow \beta$ on them. Now, since $I$ is a model of $P$ and $I \models L_1, \ldots, L_n$, we obtain that $A$ is true in $I$ and by definition of $\Phi_P$ also in $\Phi_P \uparrow (\beta + 1)$.

If condition (ii) applies, then for each clause $A \leftarrow L_1, \ldots, L_n$ in ground($P$) there is some $i$ such that $l(A) > l(L_i)$ and $L_i$ is false in $I$. Hence we obtain that $L_i$ is false in both $\Phi_P \uparrow \beta$ and $I$ by the induction hypothesis and it follows that $A$ is false in $\Phi_P \uparrow (\beta + 1)$ by definition of $\Phi_P$ and also false in $I$ since $I$ is supported.

By 2), $I$ is supported. By 1), $P$ has a unique supported model. Hence $I$ is the unique supported model of $P$.

The following definition replaces the respective part of [Hit01, Definition 5.0.2].

**0.2 Definition** A normal logic program is called $\Phi$-accessible if it satisfies one of the equiv-
0.3 Remark The following condition is not equivalent to $\Phi$-accessibility: There exists a model $I$ and a level mapping $l$ such that $I$ is a model of $P$ whose restriction to the predicate symbols in $\text{Neg}_{\Phi}^*$ is a supported model of $P^-$, and each $A \in B_P$ satisfies either (i) or (ii) from 2) above.

**Proof:** The following program is a counterexample:

\[
p \leftarrow q \\
q \leftarrow r \\
q \leftarrow p
\]

It satisfies the above conditions for the model $I = \{p, q, r\}$ and the level mapping $l(p) = 2 > l(q) = 1 > l(r) = 0$. The program has no total Fitting semantics. ■

0.4 Remark (Heinze) The following condition is not equivalent to $\Phi$-accessibility: There exists a model $I$ and a level mapping $l$ such that $I$ is a model of $P$ and each $A \in B_P$ satisfies either (i) from 2) above or (iv).

(iv) For each clause $A \leftarrow L_1, \ldots, L_n$ in $\text{ground}(P)$ with head $A$ there exists $i$ such that $I \not\models L_i$, $I \not\models A$ and $l(A) > l(L_i)$.

**Proof:** The following program is a counterexample.

\[
p \leftarrow \neg p, \neg q
\]

It satisfies the above conditions for the model $I = \{q\}$ and the level mapping $l(p) = 1 > l(q) = 0$. The program has no total Fitting semantics.

Note that the program from Remark 0.3 also serves as a counterexample. ■

We note that the proof of [Hit01, Proposition 5.5.3] can be carried over using that $I$ is supported. We repeat it for convenience.

0.5 Proposition Let $P$ be $\Phi$-accessible. Then $T_P$ is strictly contracting with respect to $\rho$.

**Proof:** Let $J, K \in I_P$ and assume that $\rho(J, K) = 2^{-\alpha}$. Then $J, K, I$ agree on all ground atoms of level less than $\alpha$. We show that $T_P(J)$ and $I$ agree on all ground atoms of level less than or equal to $\alpha$. A similar argument shows that $T_P(K)$ and $I$ agree on all ground atoms of level less than or equal to $\alpha$, and this suffices.

Let $A \in T_P(J)$ with $l(A) \leq \alpha$. Then there must be a clause $A \leftarrow L_1, \ldots, L_n$ in $\text{ground}(P)$ such that $J \models L_1 \land \cdots \land L_n$. Since $I$ and $J$ agree on all ground atoms of level less than $\alpha$, condition (ii) from Theorem 0.1 2) cannot hold, because if $I \not\models L_i$ with $l(A) > l(L_i)$, then $J \not\models L_i$ and consequently $J \not\models L_1 \land \cdots \land L_n$, which is a contradiction. Therefore, condition (i) of Theorem 0.1 2) holds and so $A \in T_P(I)$. Since $I$ is supported and $T_P(I) = I$ we conclude $A \in I$.  

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Conversely, suppose that $A \in I$. Since $I = T_P(I)$, there must be a clause $A \leftarrow L_1, \ldots, L_n$ in $\text{ground}(P)$ such that $I \models L_1 \land \cdots \land L_n$. Thus, condition (i) of Theorem 0.1.2 must hold, and so we can assume that $A \leftarrow L_1, \ldots, L_n$ also satisfies $l(A) > l(L_i)$ for $i = 1, \ldots, n$. Since $I$ and $J$ agree on all ground atoms of level less than $\alpha$, we have $J \models L_1 \land \cdots \land L_n$ and hence $A \in T_P(J)$ as required.

Finally, we note that the proof of [Hit01, Theorem 8.2.2] is unaffected.

References