Reasoning in Circumscriptive $\mathcal{ALCO}$

Stephan Grimm[1], Pascal Hitzler[2]

[1] FZI Research Center for Information Technologies, University of Karlsruhe, Germany
grimm@fzi.de
[2] Institute AIFB, University of Karlsruhe, Germany
hitzler@aifb.uni-karlsruhe.de

Abstract. Non-monotonic extensions of description logics (DLs) allow for default and local closed-world reasoning and are an acknowledged desired feature for applications, e.g. in the Semantic Web. A recent approach to such an extension is based on McCarthy's circumscription, which rests on the principle of minimising the extension of selected predicates to locally close off dedicated parts of a domain model. While decidability and complexity results have been established in the literature, no practical algorithmisation for circumscriptive DLs has been proposed so far. In this paper, we present a tableaux calculus that can be used as a sound and complete decision procedure for concept satisfiability with respect to concept-circumscribed $\mathcal{ALCO}$ knowledge bases. The calculus builds on existing tableaux for classical DLs, extended by the notion of a preference clash to detect the non-minimality of constructed models.

1 Introduction

Modern description logics (DLs) are formalisations of semantic networks and frame-based knowledge representation systems that build on classical logic. However, to also capture non-classical features, such as default and local closed-world reasoning, non-monotonic extensions to DLs have been investigated. While in the past such extensions were primarily devised on top of autoepistemic operators [7] and default inclusions [2], a recent proposal [4] is to extend DLs by circumscription and to perform non-monotonic reasoning on circumscribed DL knowledge bases. In circumscription, the extension of selected predicates – i.e. concepts or roles in the DL case – can be explicitly minimised to locally close off dedicated parts of a domain model, which results in a default reasoning behaviour. In contrast to the former approaches, non-monotonic reasoning in circumscriptive DLs also applies to "unknown individuals" not explicitly mentioned in a knowledge base, but whose existence is guaranteed due to existential quantification.

The proposal in [4] presents a semantics for circumscriptive DLs together with decidability and complexity results, in particular for fragments of the logic $\mathcal{ALCQIO}$. However, a practical algorithmisation for reasoning in circumscriptive DLs has not been addressed so far. In this paper, we present such an algorithmisation that builds on existing DL tableaux methods. In particular, we present a tableaux calculus that supports reasoning with concept-circumscribed knowledge bases in the logic $\mathcal{ALCO}$. We focus on the reasoning task of concept satisfiability, which is motivated by an application of non-monotonic reasoning in a Semantic Web setting, described in [9]. While typical examples in the circumscription literature deal with defeasible conclusions of circumscriptive abnormality theories, in this setting we use minimisation of concepts to realise a local closed-world assumption for the matchmaking of semantically annotated resources.

The reason for our choice of $\mathcal{ALCO}$ as the underlying DL is twofold. First, we want to present the circumscriptive extensions for the simplest expressive DL $\mathcal{ALC}$ for sake of a clear and concise description of the tableaux modifications. Second, there is the necessity to deal with nominals within the calculus in order to keep track of extensions of minimised concepts, so we include $\mathcal{O}$.

The basic idea behind our calculus is to detect the non-minimality of candidate models, produced by a tableaux procedure for classical DLs, via the notion of a preference clash, and based on the construction of a classical DL knowledge base that has a model if and only if the original candidate model produced is not minimal. This check can be realised by reasoning in classical DLs with nominals and equality between individuals. We formally prove this calculus to be a sound and complete decision procedure for concept satisfiability in circumscriptive $\mathcal{ALCO}$. A similar idea has been applied in [12] for circumscriptive reasoning in first-order logic, where a tableaux for first-order formulas in clausal form was presented. However, this calculus does not directly yield a decision procedure for reasoning with DLs as it is only decidable if function symbols are disallowed, which correspond to existential restrictions in DLs.

The report is structured as follows. In Section 2 we recall circumscriptive description logics from [4] for the case of $\mathcal{ALCO}$. In Section 3, we present our tableaux calculus and prove it to be a decision procedure for circumscriptive $\mathcal{ALCO}$. We conclude in Section 4.

2 Description Logics and Circumscription

Description Logics (DLs) [1] are typically fragments of first-order predicate logic that provide a well-studied formalisation for knowledge representation systems. Circumscription [11], on the other hand, is an approach to non-monotonic reasoning based on the explicit minimisation of selected predicates. In this section, we present the description logic $\mathcal{ALCO}$ extended with circumscription according to [4], which allows for non-monotonic reasoning with DL knowledge bases.

2.1 Circumscriptive $\mathcal{ALCO}$

The basic elements to represent knowledge in DLs are *individuals* that represent concrete objects within a domain of discourse, *concepts* that group together individuals with common properties, and *roles* that put individuals in relation. The countably infinite sets N_I, N_C and N_r of individual names, concept names and role names, respectively, form the basis to construct the syntactic elements of $\mathcal{ALCO}$ according to the following grammar, in which $A \in \mathsf{N}_C$ denotes an atomic concept, C, C_i denote complex concepts, $r \in \mathsf{N}_r$ denotes a role and $a_i \in \mathsf{N}_I$ denote individuals.

$$C, D \longrightarrow \bot \mid \top \mid A \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid \exists\, r\,.C \mid \forall\, r\,.C \mid \{a_1, \ldots, a_n\}$$

The semantics of the syntactic elements of $\mathcal{ALCO}$ is defined in terms of an *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ with a non-empty set $\Delta^{\mathcal{I}}$ as the *domain* and an *interpretation function* $\cdot^{\mathcal{I}}$ that maps each individual $a \in \mathsf{N}_I$ to a distinct element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ and that interprets (possibly) complex concepts and roles as follows.

$$\begin{aligned}
\top^{\mathcal{I}} = \Delta^{\mathcal{I}} \ &, \ \bot^{\mathcal{I}} = \emptyset \\
A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \ &, \ r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \\
(C_1 \sqcap C_2)^{\mathcal{I}} &= C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}} \\
(C_1 \sqcup C_2)^{\mathcal{I}} &= C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\forall\, r\,.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \forall y.(x,y) \in r^{\mathcal{I}} \rightarrow y \in C^{\mathcal{I}}\} \\
(\exists\, r\,.C)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} \mid \exists y.(x,y) \in r^{\mathcal{I}} \wedge y \in C^{\mathcal{I}}\} \\
(\{a_1, \ldots, a_n\})^{\mathcal{I}} &= \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}
\end{aligned}$$

Notice that we assume unique names for individuals, i.e. $a_1^{\mathcal{I}} \neq a_2^{\mathcal{I}}$ for any interpretation $\mathcal{I}$ and any pair $a_1, a_2 \in \mathsf{N}_I$.

An $\mathcal{ALCO}$ knowledge base *KB* is a set of *axioms* that are formed by concepts, roles and individuals. A *concept assertion* is an axiom of the form $C(a)$ that assigns the membership of an individual a to a concept C. A *role assertion* is an axiom of the form $r(a_1, a_2)$ that assigns a directed relation between two individuals a_1, a_2 by the role r. A *concept inclusion* is an axiom of the form $C_1 \sqsubseteq C_2$ that states the subsumption of the concept C_1 by the concept C_2, while a *concept equivalence* axiom $C_1 \equiv C_2$ is a shortcut for two inclusions $C_1 \sqsubseteq C_2$ and $C_2 \sqsubseteq C_1$. An interpretation $\mathcal{I}$ satisfies a concept assertion $C(a)$ if $a^{\mathcal{I}} \in C^{\mathcal{I}}$, a role assertion $r(a_1, a_2)$ if $(a_1^{\mathcal{I}}, a_2^{\mathcal{I}}) \in r^{\mathcal{I}}$, a concept inclusion $C_1 \sqsubseteq C_2$ if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ and a concept equivalence $C_1 \equiv C_2$ if $C_1^{\mathcal{I}} = C_2^{\mathcal{I}}$. An interpretation that satisfies all axioms of a knowledge base *KB* is called a *model* of *KB*. A concept C is called *satisfiable with respect to KB* if *KB* has a model in which $C^{\mathcal{I}} \neq \emptyset$ holds.

We now turn to the circumscription part of the formalism, that allows for non-monotonic reasoning by explicit minimisation of selected $\mathcal{ALCO}$ concepts. We adopt a slightly simplified form of the circumscriptive DLs presented in [4] by restricting our formalism to parallel concept circumscription (without prioritisation among minimised concepts). For this purpose we define the notion of a *circumscription pattern* as follows.

Definition 1 (circumscription pattern, $<_{\mathsf{CP}}$). *A* circumscription pattern[3] CP *is a tuple* (M, F, V) *of sets of atomic concepts called the* minimised, fixed *and* varying *concepts. Based on* CP*, a preference relation on interpretations is defined by setting* $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ *if and only if the following conditions hold:*

(i) $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ *and* $a^{\mathcal{J}} = a^{\mathcal{I}}$ *for all* $a^{\mathcal{J}} \in \Delta^{\mathcal{J}}$
(ii) $\bar{A}^{\mathcal{J}} = \bar{A}^{\mathcal{I}}$ *for all* $\bar{A} \in F$
(iii) $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$ *for all* $\tilde{A} \in M$
(iv) there is an $\tilde{A} \in M$ *such that* $\tilde{A}^{\mathcal{J}} \subset \tilde{A}^{\mathcal{I}}$

For non-monotonic reasoning, a classical $\mathcal{ALCO}$ knowledge base is circumscribed with a circumscription pattern and reasoning is performed by means of the resulting *circumscribed knowledge base*, defined as follows.

Definition 2 (circumscribed knowledge base). *A* circumscribed knowledge base $circ_{\mathsf{CP}}(KB)$ *is a knowledge base* KB *together with a circumscription pattern* $\mathsf{CP} = (M, F, V)$*, such that the sets* M*,* F *and* V *partition the atomic concepts that occur in* KB*. An interpretation* $\mathcal{I}$ *is a model of* $circ_{\mathsf{CP}}(KB)$ *if* $\mathcal{I}$ *is a model of* KB *and there exists no model* $\mathcal{J}$ *of* KB *with* $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$*.*

The intuition behind the preference relation is to identify interpretations that are "smaller" in the extensions of minimised concepts than others, to select only the "smallest" ones as the *preferred* models of a knowledge base. Fixed concepts can be used to restrict this selection and to prevent certain models from being preferred.

2.2 Reasoning with Circumscribed Knowledge Bases

The typical DL reasoning tasks are defined as expected (see [4]) with respect to the models of a circumscribed knowledge base $\mathrm{circ}_{\mathsf{CP}}(KB)$, which are just the preferred models of KB with respect to CP. For our calculus, we focus on concept satisfiability, which we define next. Other reasoning tasks can be reduced to concept satisfiability, as described in [4].

Definition 3 (circumscriptive concept satisfiability). *A concept* C *is* satisfiable with respect to a circumscribed knowledge base $\mathrm{circ}_{\mathsf{CP}}(KB)$ *if some model* $\mathcal{I}$ *of* $circ_{\mathsf{CP}}(KB)$ *satisfies* $C^{\mathcal{I}} \neq \emptyset$*.*

Observe that in classical DLs an atomic concept A is satisfiable with respect to a knowledge base KB "by default" if there is no evidence for its unsatisfiability in KB, i.e. any A is satisfiable with respect to the empty knowledge base. Now suppose that A is a minimised concept in a circumscription pattern CP by which KB is circumscribed. Then, A is unsatisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB = \emptyset)$. Only if we explicitly assure that the extension of A is non-empty, e.g. by setting $KB = \{A(a)\}$, A becomes satisfiable.

A known result in circumscription is that there is a close relation between fixed and minimised predicates. Namely, fixed concepts are simulated by minimising them together with their complements, which is achieved through introducing additional concept names and respective equivalence axioms. The proofs of Lemma 1 and Proposition 1 are similar to that of [6, Theorem 1] but have been adapted from the first-order case to the setting of circumscriptive description logics.

Lemma 1 (coincidence of complementarily minimised concepts). *Let* $\tilde{A}$*,* $\tilde{B}$ *be atomic concepts and let* KB *be a knowledge base with* $KB \models \tilde{A} \equiv \neg\tilde{B}$*. Furthermore, let* $\mathsf{CP} = (M, F, V)$ *be a circumscription pattern with* $\tilde{A}, \tilde{B} \in M$*. For any two models* $\mathcal{I}$*,*$\mathcal{J}$ *of* KB*,* $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ *implies both* $\tilde{A}^{\mathcal{J}} = \tilde{A}^{\mathcal{I}}$ *and* $\tilde{B}^{\mathcal{J}} = \tilde{B}^{\mathcal{I}}$*.*

Proof. Let $\mathcal{I}$,$\mathcal{J}$ be models of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. According to condition (iii) from Definition 1, this implies a) $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$ and b) $\tilde{B}^{\mathcal{J}} \subseteq \tilde{B}^{\mathcal{I}}$, as both $\tilde{A}$ and $\tilde{B}$ are minimised. Since $\mathcal{I}$ and $\mathcal{J}$ are models of KB, the extension of $\tilde{A}$ is just the complement of the extension of $\tilde{B}$ in both $\mathcal{I}$ and $\mathcal{J}$. Hence, a) implies $\tilde{B}^{\mathcal{J}} \supseteq \tilde{B}^{\mathcal{I}}$ and b) implies $\tilde{A}^{\mathcal{J}} \supseteq \tilde{A}^{\mathcal{I}}$, such that we have both $\tilde{A}^{\mathcal{J}} = \tilde{A}^{\mathcal{I}}$ and $\tilde{B}^{\mathcal{J}} = \tilde{B}^{\mathcal{I}}$. □

[3] The notion of circumscription pattern introduced in [4] is more general and allows the sets M, F and V to also contain roles. There, a circumscription pattern according to Definition 1 is called a concept circumscription pattern. However, in the general case role circumscription leads to undecidability, which was also shown in [4]. As the calculus presented in this report does not allow for role circumscription, we use the term circumscription pattern to denote a concept circumscription pattern in the sense of [4].

Proposition 1 (simulation of concept fixation). *Let C be a concept, let KB be a knowledge base and let $\mathsf{CP} = (M, F, V)$ be a circumscription pattern with $F = \{\bar{A}_1, \ldots, \bar{A}_n\}$. Furthermore, let*

$$KB' = KB \cup \{\tilde{A}_i \equiv \neg\bar{A}_i \mid 1 \leq i \leq n\}$$

and let

$$\mathsf{CP}' = (M \cup \{\tilde{A}_1, \ldots, \tilde{A}_n, \bar{A}_1, \ldots, \bar{A}_n\}, \emptyset, V)\ ,$$

where $\tilde{A}_1, \ldots, \tilde{A}_n$ are atomic concepts that do not occur in KB, CP or C. Then, C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$ if and only if it is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}'}(KB')$.

Proof. Assume that C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$ and let $\mathcal{I}$ be a model of $\mathrm{circ}_{\mathsf{CP}}(KB)$ with $C^{\mathcal{I}} \neq \emptyset$. Let $\mathcal{I}'$ be an interpretation that coincides with $\mathcal{I}$ except that $\tilde{A}_i^{\mathcal{I}'} = \Delta^{\mathcal{I}} \setminus \bar{A}_i^{\mathcal{I}}$ for $i = 1, \ldots, n$. Since none of the $\tilde{A}_i$ occurs in KB, $\mathcal{I}'$ is a model of KB', and since $C^{\mathcal{I}} = C^{\mathcal{I}'} \neq \emptyset$, C is satisfiable with respect to KB'. To prove that C is also satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}'}(KB')$, we show by contradiction that $\mathcal{I}'$ is minimal with respect to $<_{\mathsf{CP}'}$.

Assume that $\mathcal{I}'$ is not minimal with respect to $<_{\mathsf{CP}'}$, such that there is a model $\mathcal{J}'$ of $\mathrm{circ}_{\mathsf{CP}'}(KB')$ with $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$. By repeated application of Lemma 1, we know that $\bar{A}_i^{\mathcal{J}'} = \bar{A}_i^{\mathcal{I}'}$ and $\tilde{A}_i^{\mathcal{J}'} = \tilde{A}_i^{\mathcal{I}'}$ hold for $i = 1, \ldots, n$. Hence, the minimised predicate $\tilde{A}$ for that $\tilde{A}^{\mathcal{J}'} \subset \tilde{A}^{\mathcal{I}'}$ holds according to condition (iv) of Definition 1 must be in the set M of the original circumscription pattern CP. Let $\mathcal{J}$ be the interpretation that coincides with $\mathcal{J}'$ except that $\tilde{A}_i^{\mathcal{J}} = \emptyset$ for $i = 1, \ldots, n$. Obviously, $\mathcal{J}$ is a model of KB. We show that $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ holds by verifying the conditions (i)-(iv) from Definition 1. Condition (i) is trivially satisfied. From $\bar{A}_i^{\mathcal{J}'} = \bar{A}_i^{\mathcal{I}'}$ it follows that $\bar{A}_i^{\mathcal{J}} = \bar{A}_i^{\mathcal{I}}$ for $i = 1, \ldots, n$, and hence condition (ii) is satisfied. Since $\tilde{A}^{\mathcal{J}'} \subseteq \tilde{A}^{\mathcal{I}'}$ holds for all $\tilde{A} \in M$, condition (iii) is satisfied. Since $\tilde{A}^{\mathcal{J}'} \subset \tilde{A}^{\mathcal{I}'}$ holds for some $\tilde{A} \in M$, also condition (iv) is satisfied. Thus, we have that $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ and $\mathcal{I}$ cannot be minimal with respect to CP, which contradicts the existence of $\mathcal{J}'$.

For the converse, assume that C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}'}(KB')$ and let $\mathcal{I}'$ be a model of $\mathrm{circ}_{\mathsf{CP}'}(KB')$ with $C^{\mathcal{I}'} \neq \emptyset$. Let $\mathcal{I}$ be an interpretation that coincides with $\mathcal{I}'$ except that $\tilde{A}_i^{\mathcal{I}} = \emptyset$ for $i = 1, \ldots, n$. Since none of the $\tilde{A}_i$ appears in either KB or C, $\mathcal{I}$ is a model of KB and $C^{\mathcal{I}} \neq \emptyset$, such that C is satisfied in $\mathcal{I}$. To prove that C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$, we show by contradiction that $\mathcal{I}$ is minimal with respect to $<_{\mathsf{CP}}$.

Assume that $\mathcal{I}$ is not minimal with respect to $<_{\mathsf{CP}}$, such that there is a model $\mathcal{J}$ of $\mathrm{circ}_{\mathsf{CP}}(KB)$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. Let $\mathcal{J}'$ be the interpretation that coincides with $\mathcal{J}$ except that $\tilde{A}^{\mathcal{J}'} = \Delta^{\mathcal{J}} \setminus \bar{A}^{\mathcal{J}}$. Obviously, $\mathcal{J}'$ is a model of KB'. We show that $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$ holds by verifying the conditions (i)-(iv) from Definition 1. Conditions (i) and (ii) are trivially satisfied. From $\bar{A}_i^{\mathcal{J}} = \bar{A}_i^{\mathcal{I}}$ we know that $\bar{A}_i^{\mathcal{J}'} = \bar{A}_i^{\mathcal{I}'}$ and also that $\tilde{A}_i^{\mathcal{J}'} = \tilde{A}_i^{\mathcal{I}'}$ for $i = 1, \ldots, n$, as any two $\bar{A}_i, \tilde{A}_i$ have complementary extensions, and since $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$ for $\tilde{A} \in M$ due to $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, condition (iii) is also satisfied. From $\tilde{A}^{\mathcal{J}} \subset \tilde{A}^{\mathcal{I}}$ for some $\tilde{A} \in M$ it follows that $\tilde{A}^{\mathcal{J}'} \subset \tilde{A}^{\mathcal{I}'}$, satisfying condition (iv). Thus, we have that $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$ and $\mathcal{I}'$ cannot be minimal with respect to CP', which contradicts the existence of $\mathcal{J}$. ◻

To illustrate the reasoning task of checking concept satisfiability with respect to circumscribed knowledge bases we present the following example.

Example 1. The following knowledge base describes species living in the arctic sea.

$$KB_1 = \{\ \mathit{Bears}(\mathit{PolarBear}),\ \neg\mathit{Bears}(\mathit{BlueWhale}),\ \mathit{EndangeredSpecies}(\mathit{BlueWhale})\ \}$$

According to KB_1, the polar bear is a kind of bear, while the blue whale is not. Moreover, the blue whale is explicitly listed to be an endangered species, while the polar bear does not occur on this list. The following circumscription pattern allows to "switch off" the open-world assumption for the list of endangered species by minimising the extension of the concept *EndangeredSpecies*.

$$\mathsf{CP} = (M = \{\mathit{EndangeredSpecies}\}, F = \emptyset, V = \{\mathit{Bears}\})$$

The concept $\mathit{Bears} \sqcap \mathit{EndangeredSpecies}$ is unsatisfiable with respect to the circumscribed knowledge base $\mathrm{circ}_{\mathsf{CP}}(KB_1)$, reflecting that there cannot be an individual that is both an endangered species and a kind of bear. The only endangered species in the preferred models of KB_1 is the blue whale, which is explicitly said to be no kind of bear.

Recently, however, the polar bear unfortunately had to be included in the list of endangered species, which is reflected by the following update of KB_1.

$$KB_2 = KB_1 \cup \{ \ \mathit{EndangeredSpecies}(\mathit{PolarBear}) \ \}$$

With respect to $\mathrm{circ}_{\mathsf{CP}}(KB_2)$, the concept $\mathit{Bears} \sqcap \mathit{EndangeredSpecies}$ is satisfiable, as the polar bear is a kind of bear and at the same time an endangered species in the preferred models of KB_2.

Instead of using a concept assertion for the explicitly mentioned individual *PolarBear*, we could alternatively update KB_1 by introducing an existentially quantified object through an inclusion axiom stating that the arctic sea is a habitat for an endangered bear species, as follows.

$$KB_3 = KB_1 \cup \{ \ \exists \mathit{isHabitatFor}.\mathit{Bears} \sqcap \mathit{EndangeredSpecies}(\mathit{ArcticSea}) \ \}$$

the concept $\mathit{Bears} \sqcap \mathit{EndangeredSpecies}$ is also satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB_3)$. Observe that in any preferred model of KB_3 the extension of *EndangeredSpecies* contains an unknown individual whose existence is propagated from the known individual *ArcticSea* via the role *isHabitatFor*. Alternative approaches to non-monotonic reasoning in DL, such as [8, 2], typically treat unknown objects differently and do not allow for this kind of reasoning.

3 Tableaux Calculus for Circumscriptive $\mathcal{ALCO}$

In this section, we introduce a tableaux calculus that decides the satisfiability of a concept with respect to a circumscribed knowledge base. We build on the notion of constraint systems, which map to tableaux branches in tableaux calculi, and we keep the presentation similar to the related work in [7] and [5].

3.1 Constraint Systems and their Solvability

In addition to the alphabet of individuals N_I, we introduce a set N_V of variable symbols. We denote elements of N_I by a, elements of N_V by x and elements of $\mathsf{N}_I \cup \mathsf{N}_V$ by o, all possibly with an index. A *constraint* is a syntactic entity of one of the forms

$$o : C \quad , \qquad (o_1, o_2) : r, \qquad \forall x.x : C$$

where C is an $\mathcal{ALCO}$ concept, r is a role and the o's are objects in $\mathsf{N}_I \cup \mathsf{N}_V$. A *constraint system*, denoted by S, is a finite set of constraints. Let N_I^S denote the individuals and let N_V^S be the variables that occur in a constraint system S, respectively.

Given an interpretation $\mathcal{I}$, we define an *$\mathcal{I}$-assignment* as a function $\alpha^{\mathcal{I}} : \mathsf{N}_I \cup \mathsf{N}_V \mapsto \Delta^{\mathcal{I}}$, that maps every variable of N_V to an element of $\Delta^{\mathcal{I}}$ and every individual a to $a^{\mathcal{I}}$, i.e. $\alpha^{\mathcal{I}}(a) = a^{\mathcal{I}}$ for all $a \in \mathsf{N}_I$.

A pair $(\mathcal{I}, \alpha^{\mathcal{I}})$ of an interpretation $\mathcal{I}$ and an $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$ *satisfies* a constraint $o : C$ if $\alpha^{\mathcal{I}}(o) \in C^{\mathcal{I}}$, a constraint $(o_1, o_2) : r$ if $(\alpha^{\mathcal{I}}(o_1), \alpha^{\mathcal{I}}(o_2)) \in r^{\mathcal{I}}$ and a constraint $\forall x.x : C$ if $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$. A *solution* for a constraint system S is a pair $(\mathcal{I}, \alpha^{\mathcal{I}})$ of an interpretation $\mathcal{I}$ and an $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$ that satisfies all constraints in S.

We denote by $S[o_1/o_2]$ the constraint system that is obtained by replacing any occurrence of object o_1 by object o_2 in every constraint in S. Furthermore, we define the constraint system S_{KB} to be obtained from an $\mathcal{ALCO}$ knowledge base KB by including one constraint of the form $a : C$ for each concept assertion $C(a) \in KB$, one constraint $(a_1, a_2) : r$ for each role assertion $r(a_1, a_1) \in KB$ and one constraint $\forall x.x : \neg C_1 \sqcup C_2$ for each concept inclusion $C_1 \sqsubseteq C_2 \in KB$, such that S_{KB} captures all the information in KB. We assume all complex concepts that occur in KB to be in negation normal form, as described in [15].

To ensure termination of our calculus in the presence of general inclusion axioms, we further need to introduce the notion of blocking (see e.g. [5]). We say that an object o_1 is a *direct predecessor* of an object o_2 if the respective constraint system S contains a role constraint $(o_1, o_2) : r$ for some role r. We denote by *predecessor* the transitive closure of the direct predecessor relation. Moreover, we say that, in a constraint system S, an object o_2 is *blocked by* an object o_1 if o_1 is a predecessor of o_2 and if $\{C \mid o_2 : C \in S\} \subseteq \{C \mid o_1 : C \in S\}$ holds.

Due to the analogy between a constraint system and a knowledge base the following Lemma holds.

Lemma 2. *Let KB be an $\mathcal{ALCO}$ knowledge base, S be a constraint system with $S_{KB} \subseteq S$ and $\mathcal{I}$ be an interpretation. If $\mathcal{I}$ is a model of KB then, for any $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$, $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S_{KB}. Furthermore, for any solution $(\mathcal{I}, \alpha^{\mathcal{I}})$ for S, $\mathcal{I}$ is a model of KB.*

Proof. $\Rightarrow$: Assume that $\mathcal{I}$ is a model of KB. As no variables occur in S_{KB}, all its concept and role constraints are of the form $a : C$ and $(a_1, a_2) : r$, respectively. Since $\mathcal{I}$ is a model of KB, $a^{\mathcal{I}} \in C^{\mathcal{I}}$ holds for any concept constraint and $(a_1^{\mathcal{I}}, a_2^{\mathcal{I}}) \in r^{\mathcal{I}}$ for any role constraint in S_{KB}. As for any assignment $\alpha^{\mathcal{I}}(a) = a^{\mathcal{I}}$, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies every concept and role constraint in S_{KB}. Moreover, universal constraints in S_{KB} have the form $\forall x.x : \neg C_1 \sqcup C_2$. Since $\mathcal{I}$ satisfies their original concept inclusion $C_1 \sqsubseteq C_2$, all universal constraints in S_{KB} are satisfied as well due to the fact that $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ implies $(\Delta^{\mathcal{I}} \setminus C_1^{\mathcal{I}}) \cup C_2^{\mathcal{I}} = \Delta^{\mathcal{I}}$. Hence, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies all constraints in S_{KB} and is therefore a solution for it.
$\Leftarrow$: Assume that $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S. Since $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies all constraints in S and $S_{KB} \subseteq S$, it satisfies all constraints in S_{KB} in particular. Hence, it holds that $\alpha^{\mathcal{I}}(o) \in C^{\mathcal{I}}$ for every concept constraint, and $(\alpha^{\mathcal{I}}(o_1), \alpha^{\mathcal{I}}(o_2)) \in r^{\mathcal{I}}$ for every role constraint in S_{KB}. Moreover, $(\Delta^{\mathcal{I}} \setminus C_1^{\mathcal{I}}) \cup C_2^{\mathcal{I}} = \Delta^{\mathcal{I}}$ holds for every universal constraint $\forall x.x : \neg C_1 \cup C_2 \in S_{KB}$, which implies $C_1^{\mathcal{I}} \sqsubseteq C_2^{\mathcal{I}}$. Since there is a one-to-one correspondence between constraints in S_{KB} and axioms in KB, $\mathcal{I}$ satisfies every concept assertion, role assertion and concept inclusion in KB and is therefore a model of KB. □

The calculus we present is based on finding a solution for constraint systems the interpretation of which is a preferred model of an initial knowledge base with respect to a circumscription pattern. For this purpose we define the notion of solvability.

Definition 4 (CP-solvability). *A constraint system S is* CP-solvable with respect to *KB if there is a model $\mathcal{I}$ of KB and an $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$ such that $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$.*

By the next proposition, we reduce circumscriptive concept satisfiability to checking a constraint system for its solvability.

Proposition 2 (satisfiability reduction). *Let KB be an $\mathcal{ALCO}$ knowledge base, CP be a circumscription pattern and C be an $\mathcal{ALCO}$ concept. C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$ if and only if $S_{KB} \cup \{x : C\}$ is CP-solvable with respect to KB.*

Proof.
$\Rightarrow$: Since C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$, there is a model $\mathcal{I}$ of $\mathrm{circ}_{\mathsf{CP}}(KB)$ in which $C^{\mathcal{I}}$ is nonempty. Let a be an individual with $a^{\mathcal{I}} \in C^{\mathcal{I}}$. Since $\mathcal{I}$ is also a model of KB and due to Lemma 2, $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S_{KB} for any $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$. Let $\alpha_{x,a}^{\mathcal{I}}$ be an $\mathcal{I}$-assignment with $\alpha_{x,a}^{\mathcal{I}}(x) = a^{\mathcal{I}}$. Then, $(\mathcal{I}, \alpha_{x,a}^{\mathcal{I}})$ satisfies, besides the constraints in S_{KB}, also the constraint $x : C$, because of $\alpha_{x,a}^{\mathcal{I}}(x) \in C^{\mathcal{I}}$, and is therefore a solution for $S_{KB} \cup \{x : C\}$. Since there is no other model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, $S_{KB} \cup \{x : C\}$ is CP-solvable with respect to KB.
$\Leftarrow$: Since $S_{KB} \cup \{x : C\}$ is CP-solvable with respect to KB, there is a model $\mathcal{I}$ of KB and an $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$ such that $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for $S_{KB} \cup \{x : C\}$. Moreover, there exists an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ with $\alpha^{\mathcal{I}}(x) = a^{\mathcal{I}} \in C^{\mathcal{I}}$ because $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies the constraint $x : C$. By definition of CP-solvability, there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, and thus, $\mathcal{I}$ is a model of $\mathrm{circ}_{\mathsf{CP}}(KB)$ in which $C^{\mathcal{I}}$ is non-empty. Hence, C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$. □

3.2 Tableaux Expansion Rules

Constraint systems are manipulated by tableaux expansion rules, which decompose the structure of complex logical constructs or replace variables by concrete individuals. By expanding a constraint system with the resulting constraints, our calculus tries to build a model for the initial knowledge base that is represented by the constraint system. To decide the satisfiability of a concept C with respect to a circumscribed knowledge base $\mathrm{circ}_{\mathsf{CP}}(KB)$ according to Proposition 2, we initialise the calculus with the constraint system $S_{KB} \cup \{x : C\}$ and repeatedly apply the tableau rules given in Table 3.2. We assume that fixed concepts are simulated by minimising them together with their complements according to Proposition 1, such that no fixed concepts occur in CP.

$\longrightarrow_{\forall_x}$	:	**if** $\forall x.x : C \in S$ and $o : C \notin S$ for some $o \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$ **then** $S \leftarrow S \cup \{o : C\}$
$\longrightarrow_{\sqcap}$	:	**if** $o : C_1 \sqcap C_2 \in S$ and $\{o : C_1, o : C_2\} \not\subseteq S$ **then** $S \leftarrow S \cup \{o : C_1, o : C_2\}$
$\longrightarrow_{\sqcup}$	:	**if** $o : C_1 \sqcup C_2 \in S$ and $\{o : C_1, o : C_2\} \cap S = \emptyset$ **then** $S \leftarrow S \cup \{o : C_1\}$ or $S \leftarrow \{o : C_2\}$
$\longrightarrow_{\exists}$	:	**if** $o_1 : \exists\, r\,.C \in S$ and $\{(o_1, o_2) : r, o_2 : C\} \not\subseteq S$ and o_1 is not blocked **then** $S \leftarrow S \cup \{(o_1, x) : r, x : C\}$, with x a new variable
$\longrightarrow_{\forall}$	:	**if** $o_1 : \forall\, r\,.C \in S$ and $(o_1, o_2) : r \in S$ and $o_2 : C \notin S$ **then** $S \leftarrow S \cup \{o_2 : C\}$
$\longrightarrow_{\mathcal{O}}$	:	**if** $x : \{a_1, \ldots, a_k\} \in S$ **then** $S \leftarrow S[x/a_i]$ for any $i \in \{1, \ldots, k\} \subset \mathbb{N}$
$\longrightarrow_{<_{\mathsf{CP}}}$	:	**if** $x : \tilde{A} \in S$ and $\tilde{A} \in M_{KB}$ **then** $S \leftarrow S[x/a]$ for any $a \in \mathsf{N}_I^S \cup \{\iota\}$, with ι a new individual

Table 1. Tableau Expansion Rules for Circumscriptive $\mathcal{ALCO}$

Observe that the rules are parametric with respect to KB and CP. We denote by F_{KB} and M_{KB} the finite projections of the potentially infinite sets F and M of fixed and minimised concepts in CP on the concepts that occur in KB.

The rules $\longrightarrow_{\forall_x}$, $\longrightarrow_{\sqcap}$, $\longrightarrow_{\exists}$ and $\longrightarrow_{\forall}$ are *deterministic* and their application yields a single resulting constraint system. Contrarily, the rules $\longrightarrow_{\sqcup}$, $\longrightarrow_{\mathcal{O}}$ and $\longrightarrow_{<_{\mathsf{CP}}}$ are *non-deterministic*, meaning that they can be applied in multiple ways that yield different constraint systems. Any such non-deterministic choice produces a branching point for backtracking when algorithmically determined. In the $\longrightarrow_{\sqcup}$-rule, the disjunction leads to the choice of expanding on either of the disjuncts, while in the $\longrightarrow_{\mathcal{O}}$- and the $\longrightarrow_{<_{\mathsf{CP}}}$-rule the presence of several individuals leads to the choice of selecting one for replacement of the variable x. Moreover, the $\longrightarrow_{<_{\mathsf{CP}}}$-rule introduces new individuals into the constraint system whenever ι is selected for replacement[4], while the $\longrightarrow_{\exists}$-rule introduces new variables whenever an object lacks a role filler.

Definition 5 (completion). *A* completion *of a constraint system S with regard to CP and KB is any constraint system that results from the application of the tableaux rules to S, using CP and KB, and to which none of the rules is applicable. (Often the parameters CP and KB for rule application are clear from the context and are omitted.)*

The repeated application of rules finally leads to a completion of the initial constraint system that contains the exhaustive decomposition of complex constraints, which is established by the following lemma.

Lemma 3 (termination). *Let S be a constraint system. A repeated application of the tableaux rules to S ultimately terminates, and yields a completion of S.*

Proof (Sketch). After application of any of the rules, S is altered such that the rule condition is not triggered again with the same parameters. Complex concepts that occur in constraints are decomposed by the rules until their finite structure finally breaks down to atomic concepts, nominals and their negations. Due to the finite structure of complex concepts and the blocking condition (see e.g. [5]) employed in the $\longrightarrow_{\exists}$-rule only finitely many variables are introduced, and whenever the $\longrightarrow_{\mathcal{O}}$-rule or the $\longrightarrow_{<_{\mathsf{CP}}}$-rule is applied to a nominal or an atomic concept, a variable is replaced by an individual, such that the number of variables cannot grow infinitely. Hence, the resulting constraint system reaches a point at which no rule is applicable to it and is then, by definition, a completion of S.

Moreover, we establish the result that the tableaux expansion rules of our calculus preserve the solvability of constraint systems as follows.

[4] The idea of including a new individual ι as a representative for the infinitely many remaining objects in $\mathsf{N}_I \setminus \mathsf{N}_I^S$ in the domain is taken from [7].

Proposition 3 (solvability preservation). *Let KB be an $\mathcal{ALCO}$ knowledge base, $\textsf{CP}$ be a circumscription pattern and S, S' be two constraint systems.*

1. *If S' is obtained from S by application of a deterministic rule then S is $\textsf{CP}$-solvable with respect to KB if and only if S' is $\textsf{CP}$-solvable with respect to KB.*
2. *If S' is obtained from S by application of a non-deterministic rule then S is $\textsf{CP}$-solvable with respect to KB if S' is $\textsf{CP}$-solvable with respect to KB. Furthermore, if S is $\textsf{CP}$-solvable with respect to KB and a non-deterministic rule applies to S then it can be applied in such a way that the resulting constraint system S' is also $\textsf{CP}$-solvable with respect to KB.*

Proof. For the first part, we consider the deterministic rules:
$\Leftarrow$: Assume that S' is obtained from S by application of a deterministic rule and that S' is $\textsf{CP}$-solvable with respect to KB. Let $(\mathcal{I}, \alpha^{\mathcal{I}})$ be a solution for S' such that $\mathcal{I}$ is a model of KB and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\textsf{CP}} \mathcal{I}$. For any of the deterministic rules it holds that S is a subset of S', and therefore $(\mathcal{I}, \alpha^{\mathcal{I}})$ is also a solution for S. Since there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\textsf{CP}} \mathcal{I}$, S is $\textsf{CP}$-solvable with respect to KB.

$\Rightarrow$: Assume that S' is obtained from S by application of a deterministic rule and that S is $\textsf{CP}$-solvable with respect to KB. Let $(\mathcal{I}, \alpha^{\mathcal{I}})$ be a solution for S such that $\mathcal{I}$ is a model of KB and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\textsf{CP}} \mathcal{I}$. We subsequently consider the various deterministic rules.
$\longrightarrow_{\forall_x}$:
If the $\longrightarrow_{\forall_x}$-rule has been applied then S contains a constraint of the form $\forall x.x : C$ and the resulting constraint system S' contains the concept constraint $o : C$ for some $o \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$. As a solution for S, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies $\forall x.x : C$ and we have that $C^{\mathcal{I}} = \Delta^{\mathcal{I}}$. From $(\mathsf{N}_I^S \cup \mathsf{N}_V^S)^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ it follows that $\alpha^{\mathcal{I}}(o) \in \Delta^{\mathcal{I}}$, and hence this concept constraint is also satisfied by $(\mathcal{I}, \alpha^{\mathcal{I}})$, which is therefore a solution for S'.
$\longrightarrow_{\sqcap}$:
If the $\longrightarrow_{\sqcap}$-rule has been applied then S contains a constraint of the form $o : C \sqcap D$ and S' contains $\{o : C, o : D\}$. As a solution for S, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies the constraint $o : C \sqcap D$, and we have that $\alpha^{\mathcal{I}}(o) \in (C \sqcap D)^{\mathcal{I}}$ and, due to the semantics of $\mathcal{ALCO}$, both $\alpha^{\mathcal{I}}(o) \in C^{\mathcal{I}}$ and $\alpha^{\mathcal{I}}(o) \in D^{\mathcal{I}}$ hold. Hence, $(\mathcal{I}, \alpha^{\mathcal{I}})$ also satisfies the two constraints $o : C$ and $o : D$, and is therefore a solution for S'.
$\longrightarrow_{\exists}$:
If the $\longrightarrow_{\exists}$-rule has been applied then S contains a constraint of the form $o : \exists\, r\,.C$ and S' contains the two constraints $(o, x) : r$ and $x : C$, where $x \notin \mathsf{N}_V^S$ is a new variable. As a solution for S, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies the constraint $o : \exists\, r\,.C$, and we have that $\alpha^{\mathcal{I}}(o) \in (\exists\, r\,.C)^{\mathcal{I}}$. Due to the semantics of $\mathcal{ALCO}$ there is some $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for which both $(\alpha^{\mathcal{I}}(o), a^{\mathcal{I}}) \in r^{\mathcal{I}}$ and $a^{\mathcal{I}} \in C^{\mathcal{I}}$ hold. Since x is new to S, we can safely assume that $\alpha^{\mathcal{I}}(x) = a^{\mathcal{I}}$. Then, $(\mathcal{I}, \alpha^{\mathcal{I}})$ also satisfies the two constraints $(o, x) : r$ and $x : C$, and is therefore a solution for S'.
$\longrightarrow_{\forall}$:
If the $\longrightarrow_{\forall}$-rule has been applied then S contains two constraints $o_1 : \forall\, r\,.C$ and $(o_1, o_2) : r$, while the resulting constraint system S' differs by $S' \setminus S = \{o_2 : C\}$. Since $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies S, it also satisfies $o_1 : \forall\, r\,.C$ and $(o_1, o_2) : r$, and we have that $\alpha^{\mathcal{I}}(o_1) \in (\forall\, r\,.C)^{\mathcal{I}}$ and $(\alpha^{\mathcal{I}}(o_1), \alpha^{\mathcal{I}}(o_2)) \in r^{\mathcal{I}}$. Due to the semantics of $\mathcal{ALCO}$, this implies that $\alpha^{\mathcal{I}}(o_2) \in C^{\mathcal{I}}$, and thus, $(\mathcal{I}, \alpha^{\mathcal{I}})$ also satisfies $o_2 : C$ and is therefore a solution for S'.

Since in all of the cases $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S' and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\textsf{CP}} \mathcal{I}$ by assumption, S' is $\textsf{CP}$-solvable with respect to KB.

For the second part, we consider the non-deterministic rules:
$\Leftarrow$: Assume that S' is obtained from S by application of a non-deterministic rule and that S' is $\textsf{CP}$-solvable with respect to KB. Let $(\mathcal{I}, \alpha^{\mathcal{I}})$ be a solution for S' such that $\mathcal{I}$ is a model of KB and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\textsf{CP}} \mathcal{I}$. We subsequently consider the various non-deterministic rules.
$\longrightarrow_{\sqcup}$:
If the $\longrightarrow_{\sqcup}$-rule has been applied then S contains a constraint of the form $o : C \sqcup D$ and the resulting constraint system S' differs from S by either $S' \setminus S = \{o : C\}$ or $S' \setminus S = \{o : D\}$. without loss of generality we can assume that $S' \setminus S = \{o : C\}$. Since $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S', it also satisfies $o : C$, and we have that $\alpha^{\mathcal{I}}(o) \in C^{\mathcal{I}}$, and thus also $\alpha^{\mathcal{I}}(o) \in C^{\mathcal{I}} \cup D^{\mathcal{I}}$. Hence, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies $o : C \sqcup D$ and is therefore a solution for S.

$\longrightarrow_{\mathcal{O}}, \longrightarrow_{<_{\mathsf{CP}}}$:
If either the $\longrightarrow_{\mathcal{O}}$-rule or the $\longrightarrow_{<_{\mathsf{CP}}}$-rule has been applied then $S' = S[x/a]$ for some individual $a \in \mathsf{N}_I$. As a solution for S', $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies all the constraints in $S[x/a]$, in particular those in which x has been replaced by a. Let $\alpha^{\mathcal{I}}_{x,a}$ be the $\mathcal{I}$-assignment that coincides with $\alpha^{\mathcal{I}}$ except that $\alpha^{\mathcal{I}}_{x,a}(x) = a^{\mathcal{I}}$. Then, $(\mathcal{I}, \alpha^{\mathcal{I}}_{x,a})$ satisfies all the constraints in S in which x occurs, and since S and S' differ only by these, also all remaining constraints in S. Hence, $(\mathcal{I}, \alpha^{\mathcal{I}}_{x,a})$ is a solution for S.

Since in all the cases there is a solution for S and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by assumption, S is CP-solvable with respect to KB.

$\Rightarrow$: Assume that S' is obtained from S by application of a non-deterministic rule and that S is CP-solvable with respect to KB. Let $(\mathcal{I}, \alpha^{\mathcal{I}})$ be a solution for S such that $\mathcal{I}$ is a model of KB and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. We subsequently consider the various non-deterministic rules.
$\longrightarrow_{\sqcup}$:
If the $\longrightarrow_{\sqcup}$-rule has been applied then S contains a constraint of the form $o : C \sqcup D$ and the resulting constraint system S' differs from S by either $S' \setminus S = \{o : C\}$ or $S' \setminus S = \{o : D\}$. Without loss of generality we can assume that $S' \setminus S = \{o : C\}$, and that $\alpha^{\mathcal{I}}(o) \in C^{\mathcal{I}}$ holds for the solution $(\mathcal{I}, \alpha^{\mathcal{I}})$ for S. Then, $(\mathcal{I}, \alpha^{\mathcal{I}})$ also satisfies $o : C$, and is therefore a solution for S'. Since there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by assumption, the $\longrightarrow_{\sqcup}$-rule can be applied to S in such a way that S' is CP-solvable with respect to KB.
$\longrightarrow_{\mathcal{O}}$:
If the $\longrightarrow_{\mathcal{O}}$-rule has been applied then S contains a constraint of the form $x : \{a_1, \ldots, a_k\}$. As a solution of S, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies this constraint and there is some individual $a_i \in \{a_1, \ldots, a_k\}$ with $\alpha(x) = a_i^{\mathcal{I}}$ that can be picked for the application of the $\longrightarrow_{\mathcal{O}}$-rule to yield $S' = S[x/a_i]$. Since the constraint $a_i : \{a_1, \ldots, a_k\}$ with $i \in \{1, \ldots, k\}$ is trivially satisfied, $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for $S[x/a_i]$, and since there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by assumption, the $\longrightarrow_{\mathcal{O}}$-rule can be applied to S in such a way that S' is CP-solvable with respect to KB.
$\longrightarrow_{<_{\mathsf{CP}}}$:
If the $\longrightarrow_{<_{\mathsf{CP}}}$-rule has been applied then S contains a constraint of the form $x : \tilde{A}$ with $\tilde{A} \in M_{\mathit{KB}}$. As a solution for S, $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies this constraint and there is some individual $a \in \mathsf{N}_I$ with $\alpha^{\mathcal{I}}(x) = a$. We distinguish the two cases in which a) a is in N_I^S and b) a is a new individual not in N_I^S:

- a) In case $a \in \mathsf{N}_I^S$, a can be picked for the application of the $\longrightarrow_{<_{\mathsf{CP}}}$-rule and it directly follows that $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for the resulting constraint system $S' = S[x/a]$.
- b) In case $a \in \mathsf{N}_I \setminus \mathsf{N}_I^S$, $\iota \in \mathsf{N}_I \setminus \mathsf{N}_I^S$ can be picked for the application of the $\longrightarrow_{<_{\mathsf{CP}}}$-rule as a representative for any new individual. Then, $S[x/a]$ and $S[x/\iota]$ differ only by the naming of an individual new to S and are in this sense isomorphic.[5] Hence, as $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for $S[x/a]$ it is also a solution for the resulting constraint system $S[x/\iota] = S'$.

Finally, since in all of the cases $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S' and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by assumption, the $\longrightarrow_{<_{\mathsf{CP}}}$-rule can be applied to S in such a way that S' is CP-solvable with respect to KB. $\square$

3.3 Notions of Clash and Detection of Inconsistencies

Once a completion of an initial constraint system has been produced, its solvability can be verified by using the notion of a *clash*. In addition to the clashes defined in [7, 14], which represent obvious contradictions in a knowledge base, we additionally introduce the notion of a *preference clash*, which reflects non-minimality of the respective model with regard to the preference relation $<_{\mathsf{CP}}$.

Definition 6 (Clashes). *Let S be a constraint system.*

S contains an inconsistency clash *if at least one of the following holds:*
(i) S contains a constraint of the form $o : \bot$.
(ii) S contains two constraints of the form $o : A$, $o : \neg A$.

S contains an individual clash *if at least one of the following holds:*

[5] See also the analogous argument in [7, Lemma 3.6].

(iii) *S contains a constraint of the form $a : \{a_1, \ldots, a_k\}$.*
with $a \neq a_i$ for all $i \in \{1, \ldots, k\} \subset \mathbb{N}$.

(vi) *S contains a constraint of the form $a : \neg\{a_1, \ldots, a_k\}$.*
with $a = a_i$ for some $i \in \{1, \ldots, k\} \subset \mathbb{N}$.

S contains a preference clash*, parameterised with a circumscription pattern CP and an $\mathcal{ALCO}$ knowledge base $K\!B$, if the following condition holds:*

(v) *the constraint system $S_{K\!B'}[\iota/x]$ has a completion, with regard to $\mathsf{CP}' = (\emptyset, \emptyset, F \cup M \cup V)$ and $K\!B'$, that does neither contain an inconsistency clash nor an individual clash, while the $\mathcal{ALCO}$ knowledge base $K\!B'$ is constructed according to Algorithm 1.*

Algorithm 1 Construct a knowledge base $K\!B'$.

Require: a constraint system S produced for an initial $\mathcal{ALCO}$ knowledge base $K\!B$ circumscribed with a circumscription pattern $\mathsf{CP} = (M, F, V)$

$D \leftarrow \{\bot\}$
$K\!B' \leftarrow K\!B$
for all $\tilde{A} \in M_{K\!B}$ **do**
  if there are constraints $a_1 : \tilde{A}, \ldots, a_n : \tilde{A} \in S$ **then**
    $K\!B' \leftarrow K\!B' \cup \{\tilde{A} \sqsubseteq \{a_1, \ldots, a_n\}\}$
    $D \leftarrow D \cup \{\{a_1, \ldots, a_n\} \sqcap \neg\tilde{A}\}$
  else
    $K\!B' \leftarrow K\!B' \cup \{\tilde{A} \sqsubseteq \bot\}$
  end if
end for
$K\!B' \leftarrow K\!B' \cup \{(\bigsqcup_{D_{\tilde{A}} \in D} D_{\tilde{A}})(\iota)\}$, with ι a new individual

---

The idea behind the construction of $K\!B'$ in Algorithm 1 is to freeze the instance situation for minimised concepts as asserted in the current constraint system perceived as reflecting some model $\mathcal{I}$ of the original knowledge base $K\!B$. Then, $K\!B'$ is constructed such that for any of its models $\mathcal{J}$ it holds that $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, and thus, checking $K\!B'$ for unsatisfiability verifies minimality of $\mathcal{I}$. By inclusion axioms for minimised concepts $\tilde{A}$, condition 3 of Definition 1 is assured to hold for each model of $K\!B'$. Moreover, by the disjunctive concept assertion, condition 4 of Definition 1 is assured to hold, such that any model of $K\!B'$ is actually “smaller” than $\mathcal{I}$ in some minimised concept, which is achieved by mapping the not uniquely named individual ι to one that already occurs in the extension of a minimised concept. Although in general we assume unique names in the formalism, the replacement of the new individual ι by the variable x within $S_{K\!B'}[\iota/x]$ in condition (v) of Definition 6 allows ι to be (indirectly) identified with some other individual.

We illustrate the detection of clashes in our calculus by means of an example.

Example 2. Consider the circumscribed knowledge base $\mathrm{circ}_{\mathsf{CP}}(K\!B)$ with the following knowledge base $K\!B$ and circumscription pattern CP.

$$\begin{aligned} K\!B &= \{\ \neg \mathit{Bears}(\mathit{BlueWhale})\,,\ \mathit{EndangeredSpecies}(\mathit{BlueWhale})\ \} \\ \mathsf{CP} &= (M = \{\mathit{EndangeredSpecies}\}, F = \emptyset, V = \{\mathit{Bears}\}) \end{aligned}$$

We perform our calculus to check whether the concept $\mathit{Bears} \sqcap \mathit{EndangeredSpecies}$ is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(K\!B)$.

We start with the constraint system initialised as follows.

$$\begin{aligned} S_{K\!B} \cup \{x : \mathit{Bears} \sqcap \mathit{EndangeredSpecies}\} = \{\ & \mathit{BlueWhale} : \neg\mathit{Bears}\,, \\ & \mathit{BlueWhale} : \mathit{EndangeredSpecies}\,,\ x : \mathit{Bears} \sqcap \mathit{EndangeredSpecies}\ \} \end{aligned}$$

From the application of the $\longrightarrow_{\sqcap}$-rule and the non-deterministic $\longrightarrow_{<_{\mathsf{CP}}}$-rule, the following two resulting completions are produced.

$$
\begin{aligned}
S_1 = \{\ & \mathit{BlueWhale} : \neg \mathit{Bears}\,,\ \mathit{BlueWhale} : \mathit{EndangeredSpecies}\,,\ \mathit{BlueWhale} : \mathit{Bears}\ \} \\
S_2 = \{\ & \mathit{BlueWhale} : \neg \mathit{Bears}\,,\ \mathit{BlueWhale} : \mathit{EndangeredSpecies}\,, \\
& \iota_0 : \mathit{Bears}\,,\ \iota_0 : \mathit{EndangeredSpecies}\ \}
\end{aligned}
$$

The completion S_1 obviously contains an inconsistency clash, since it contains both the constraints $\mathit{BlueWhale} : \mathit{Bears}$ and $\mathit{BlueWhale} : \neg \mathit{Bears}$.

For the completion S_2, we construct the knowledge base KB' according to Algorithm 1 as follows.

$$
\begin{aligned}
\mathit{KB}' = \{\ & \neg \mathit{Bears}(\mathit{BlueWhale})\,,\ \mathit{EndangeredSpecies}(\mathit{BlueWhale})\,, \\
& \mathit{EndangeredSpecies} \sqsubseteq \{\mathit{BlueWhale}\,, \iota_0\}\,, \\
& \neg \mathit{EndangeredSpecies} \sqcap \{\mathit{BlueWhale}\,, \iota_0\}(\iota) \qquad \}
\end{aligned}
$$

To check whether the completion S_2 contains a preference clash, we need to check if KB' has a model, while the new individual ι is not uniquely named. In our calculus, this is done by trying to produce a clash-free completion of the following respective constraint system.

$$
\begin{aligned}
S_{\mathit{KB}'}[\iota/x] = \{\ & \mathit{BlueWhale} : \neg \mathit{Bears}\,,\ \mathit{BlueWhale} : \mathit{EndangeredSpecies}\,, \\
& \forall x.x : \neg \mathit{EndangeredSpecies} \sqcup \{\mathit{BlueWhale}, \iota_0\}\,, \\
& x : \neg \mathit{EndangeredSpecies} \sqcap \{\mathit{BlueWhale}\,, \iota_0\} \qquad \}
\end{aligned}
$$

The constraint system $S_{\mathit{KB}'}[\iota/x]$ has the following completion S', in which the new variable x has been replaced by the individual ι_0 that was introduced by the $\longrightarrow_{<_{\mathsf{CP}}}$-rule in the completion process for S_2 before.

$$
S' \ = \ S_{\mathit{KB}'}[\iota/x] \ \cup \ \{\ \iota_0 : \neg \mathit{EndangeredSpecies}\,,\ \iota_0 : \{\mathit{BlueWhale}, \iota_0\}\ \}
$$

S' does neither contain an inconsistency clash nor an individual clash, and thus, S_2 contains a preference clash according to condition (v) of Definition 6. Since both S_1 and S_2 contain some clash, the initial constraint system $S_{\mathit{KB}} \cup \{x : \mathit{Bears} \sqcap \mathit{EndangeredSpecies}\}$ has no clash-free completion. Hence, due to Proposition 2 the concept $\mathit{Bears} \sqcap \mathit{EndangeredSpecies}$ is unsatisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(\mathit{KB})$.

In the description logic literature [5, 14, 3, 10], tableaux methods for sound and complete reasoning have been proposed for various DL variants including $\mathcal{ALCO}$. They detect inconsistencies in DL knowledge bases by checking completions of constraint systems for the occurrence of a clash. For sake of completeness, we include this result in our presentation in form of the following proposition.

Proposition 4 ($\mathcal{ALCO}$ correctness). *Let KB be an $\mathcal{ALCO}$ knowledge base and S be the completion of a constraint system that contains at least the constraints of S_{KB}, with regard to any circumscription pattern and KB. Then S has a solution if and only if it contains neither an inconsistency clash nor an individual clash.*

Proof.
$\Rightarrow$: Assume that $(\mathcal{I}, \alpha^{\mathcal{I}})$ is a solution for S, for some model $\mathcal{I}$ of KB and an $\mathcal{I}$-assignment $\alpha^{\mathcal{I}}$. We show by contradiction that S does neither contain an inconsistency clash nor an individual clash.
–*inconsistency clash*:
Assume that S contains an inconsistency clash. Then, it contains either a constraint $o : \bot$ or two constraints $\{o : A, o : \neg A\}$, and $\mathcal{I}$ must satisfy $\alpha^{\mathcal{I}}(o) \in A^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}) = \emptyset$. Hence, $\mathcal{I}$ cannot be a model of KB, which contradicts S to contain an inconsistency clash.
–*individual clash*:
Assume that S contains an individual clash. Then, it contains either a constraint $a : \{a_1, \ldots, a_n\}$ with $a \neq a_i$ for all $i \in \{1, \ldots, n\}$ or a constraint $a : \neg\{a_1, \ldots, a_n\}$ with $a = a_i$ for some $i \in \{1, \ldots, n\}$. In the first case $\mathcal{I}$ must satisfy $\alpha^{\mathcal{I}}(a) = a^{\mathcal{I}} = a_i^{\mathcal{I}} = \alpha^{\mathcal{I}}(a_i)$ for two distinct individuals a and a_i, which cannot be the case since we assumed unique names. In the second case $\mathcal{I}$ must satisfy $\alpha^{\mathcal{I}}(a) = a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus \{\ldots, a^{\mathcal{I}}, \ldots\}$. In both cases, $\mathcal{I}$ cannot be a model of KB, which contradicts S to contain an individual clash.

$\Leftarrow$: Let S neither contain an inconsistency clash nor an individual clash. We construct an interpretation $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ from S with $\Delta^\mathcal{I} = \mathsf{N}_I^S \cup \{\iota_x \mid x \in \mathsf{N}_V^S\}$, together with an $\mathcal{I}$-assignment $\alpha^\mathcal{I} : \mathsf{N}_I^S \cup \mathsf{N}_V^S \mapsto \Delta^\mathcal{I}$ that maps each variable x in S to a distinct individual ι_x not occurring in S, i.e. $\alpha^\mathcal{I}(x) = \iota_x$ for $x \in \mathsf{N}_V^S$.[6] We define the interpretation function $\cdot^\mathcal{I}$ such that an atomic concept A is interpreted as $A^\mathcal{I} = \{\alpha^\mathcal{I}(o) \mid o : A \in S\}$ and a role r as $r^\mathcal{I} = \{(o_1, o_2) \mid (o_1, o_2) : r \in S \text{ or } o_1 \text{ is blocked by } o_3 \text{ with } (o_3, o_2) : r \in S\}$. We show that $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies every constraint in S.

For any concept constraint $o : C$ in S we show that $\alpha^\mathcal{I}(o) \in C^\mathcal{I}$ by induction on the structure of C. As base cases, we consider concepts of the form A, $\neg A$ and $\{a_1, \ldots, a_k\}$, $\neg\{a_1, \ldots, a_k\}$, as well as roles. If $o : A$ is in S then $\alpha^\mathcal{I}(o) \in A^\mathcal{I}$ by definition. If $o : \neg A$ is in S then $o : A$ is not in S, since S does not contain an inconsistency clash, and thus $\alpha^\mathcal{I}(o) \in \Delta^\mathcal{I} \setminus A^\mathcal{I}$. If $a : \{a_1, \ldots, a_k\}$ is in S then $\alpha^\mathcal{I}(a) \in \{a_1^\mathcal{I}, \ldots, a_k^\mathcal{I}\}$, since S contains no individual clash; as the $\longrightarrow_\mathcal{O}$-rule is not applicable, $x : \{a_1, \ldots, a_k\}$ cannot be in S . If $a : \neg\{a_1, \ldots, a_k\}$ is in S then $\alpha^\mathcal{I}(a) \notin \{a_1^\mathcal{I}, \ldots, a_k^\mathcal{I}\}$ and thus $\alpha^\mathcal{I}(a) \in \Delta^\mathcal{I} \setminus \{a_1^\mathcal{I}, \ldots, a_k^\mathcal{I}\}$, since S contains no individual clash, while if $x : \neg\{a_1, \ldots, a_k\}$ is in S then $\alpha^\mathcal{I}(x) = \iota_x \in \Delta^\mathcal{I} \setminus \{a_1^\mathcal{I}, \ldots, a_k^\mathcal{I}\}$ due to the unique name assumption. If $(o_1, o_2) : r$ is in S then $(\alpha^{\mathcal{I}_S}(o_1), \alpha^{\mathcal{I}_S}(o_2)) \in r^\mathcal{I}$ by definition. For the induction step, we subsequently consider the various complex forms of concepts. Observe that none of the tableaux rules can be applied to S, as it is a completion.

- If S contains $o : C \sqcap D$ then it contains both $o : C$ and $o : D$, since the $\longrightarrow_\sqcap$-rule cannot be applied. From the induction hypothesis we know that S satisfies these two constraints, and thus, both $\alpha^\mathcal{I}(o) \in C^\mathcal{I}$ and $\alpha^\mathcal{I}(o) \in C^\mathcal{I}$ hold. Hence, $\alpha^\mathcal{I}(o) \in (C^\mathcal{I} \cap D^\mathcal{I}) = (C \sqcap D)^\mathcal{I}$ and $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies $o : C \sqcap D$.
- If S contains $o : C \sqcup D$ then it contains either $o : C$ or $o : D$, since the $\longrightarrow_\sqcup$-rule cannot be applied. Without loss of generality, we can assume that S contains $o : C$. From the induction hypothesis we know that $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies $o : C$, i.e. $\alpha^\mathcal{I}(o) \in C^\mathcal{I}$, and therefore we have that $\alpha^\mathcal{I}(o) \in (C^\mathcal{I} \cup D^\mathcal{I}) = (C \sqcup D)^\mathcal{I}$ and that $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies $o : C \sqcup D$.
- If S contains $o_1 : \exists r.C$ then, since the $\longrightarrow_\exists$-rule cannot be applied, we have one of the following two cases: a) S contains both $(o_1, o_2) : r$ and $o_2 : C$ for some $o_2 \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$, b) o_1 is blocked by some $o_3 \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$. In case a) we know from the induction hypothesis that S satisfies these two constraints, and thus, both $(\alpha^\mathcal{I}(o_1), \alpha^\mathcal{I}(o_2)) \in r^\mathcal{I}$ and $\alpha^\mathcal{I}(o_2) \in C^\mathcal{I}$ hold. Hence, $\alpha^\mathcal{I}(o_1) \in (\exists r.C)^\mathcal{I}$ and S satisfies $o_1 : \exists r.C$. In case b) we know from the subset blocking condition that S contains the constraint $o_3 : \exists r.C$, since o_1 is blocked by o_3. As o_3 cannot be blocked itself (see [5, Lemma 3.5]) and S is complete, there are constraints $(o_3, o_2) : r$, $o_2 : C$ in S for some $o_2 \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$. From the induction hypothesis we know that $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies these constraints, and thus, both $(\alpha^\mathcal{I}(o_3), \alpha^\mathcal{I}(o_2)) \in r^\mathcal{I}$ and $\alpha^\mathcal{I}(o_2) \in C^\mathcal{I}$ hold. From the definition of $\cdot^\mathcal{I}$ and the fact that o_1 is blocked by o_3 it follows that $(\alpha^\mathcal{I}(o_1), \alpha^\mathcal{I}(o_2)) \in r^\mathcal{I}$, and hence, $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies $o_1 : \exists r.C$.
- If S contains $o_1 : \forall r.C$ then it also contains $o_2 : C$ for every $o_2 \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$ with $(o_1, o_2) : r \in S$, since the $\longrightarrow_\forall$-rule cannot be applied. In case o_1 is blocked by some $o_3 \in \mathsf{N}_I^S \cup \mathsf{N}_V^S$, S contains constraints $o_2 : C$ for every o_2 with $(o_3, o_2) : r \in S$, since it contains $o_3 : \forall r.C$ due to the subset blocking condition. By the definition of $\cdot^\mathcal{I}$ and the fact that o_3 cannot be blocked itself (see [5, Lemma 3.5]), role instances $r(\alpha^\mathcal{I}(o_1), \alpha^\mathcal{I}(o_2))$ can only be induced by the role constraints $(o_1, o_2) : r$ and $(o_3, o_2) : r$, and from the induction hypothesis we know that these are satisfied by S together with the respective concept constraints $o_2 : C$. Hence, we have that $\alpha^\mathcal{I}(o_2) \in C^\mathcal{I}$ for all role instances $r(\alpha^\mathcal{I}(o_1), \alpha^\mathcal{I}(o_2))$ and S satisfies $o_1 : \forall r.C$.

By this, we have shown that $(\mathcal{I}, \alpha^\mathcal{I})$ satisfies every constraint in S and is therefore a solution for it. $\square$

Based on this correspondence between clash-free completions and their solutions, we can establish the correlation between solvability of constraint systems and the absence of preference clashes in their completions, as the main result of this report by the following proposition.

Proposition 5 (circumscriptive $\mathcal{ALCO}$ correctness). *Let KB be an $\mathcal{ALCO}$ knowledge base, CP be a circumscription pattern and S be the completion of a constraint system that contains at least the constraints of S_{KB}, with regard to CP and KB. S is CP-solvable with respect to KB if and only if it contains no inconsistency clash, no individual clash and no preference clash with respect to CP and KB.*

[6] This construction is similar to the notion of a *canonical interpretation* as used in the literature such as [5]

Proof.
$\Rightarrow$: Assume that S is CP-solvable with respect to KB. According to Definition 4 there is a solution $(\mathcal{I}, \alpha^{\mathcal{I}})$ for S, such that $\mathcal{I}$ is a model of KB and there is no model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. From Proposition 4, we know that S does neither contain an inconsistency clash nor an individual clash. We show by contradiction that S does also not contain a preference clash.

Assume that S contains a preference clash with respect to CP and KB. Then, $S_{\mathit{KB}'}[\iota/x]$ has a completion S' with regard to $\mathsf{CP} = (\emptyset, \emptyset, M \cup F \cup V)$ and KB' that contains no inconsistency and no individual clash, where the knowledge base KB' is constructed based on CP and KB according to Algorithm 1. Observe that, by construction, $\mathit{KB} \subset \mathit{KB}'$ and that ι is a new individual in KB' that cannot occur in KB. Hence, we have that $S_{\mathit{KB}} \subset S_{\mathit{KB}'}[\iota/x] \subseteq S'$. Proposition 4($\Leftarrow$) implies that there is a solution $(\mathcal{J}, \alpha^{\mathcal{J}})$ for S', since S' is clash-free. Due to Lemma 2, and since $S_{\mathit{KB}} \subseteq S'$, it follows that $\mathcal{J}$ is a model of both KB' and KB. It remains to show that $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, to contradict the containment of a preference clash in S. Without loss of generality, we can assume that $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$ and that $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all individuals $a \in \mathsf{N}_I$. We prove the following claims: a) $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$ for all $\tilde{A} \in M_{\mathit{KB}}$, and b) $\tilde{A}^{\mathcal{J}} \subset \tilde{A}^{\mathcal{I}}$ for some $\tilde{A} \in M_{\mathit{KB}}$.

- a) Due to the inclusion axioms for minimised concepts inserted into KB' by Algorithm 1, and since $\mathcal{J}$ is a model of KB', $\mathcal{J}$ has the property $\tilde{A}^{\mathcal{J}} \subseteq \{\alpha^{\mathcal{J}}(a) \mid a : \tilde{A} \in S\}$ for each $\tilde{A} \in M_{\mathit{KB}}$. For every $\tilde{A} \in M_{\mathit{KB}}$, all the constraints $a : \tilde{A} \in S$ are satisfied by $(\mathcal{I}, \alpha^{\mathcal{I}})$, i.e. $\alpha^{\mathcal{I}}(a) \in \tilde{A}^{\mathcal{I}}$, and therefore we have that $\{\alpha^{\mathcal{I}}(a) \mid a : \tilde{A} \in S\} \subseteq \tilde{A}^{\mathcal{I}}$. Since $\alpha^{\mathcal{I}}$ and $\alpha^{\mathcal{J}}$ coincide on individuals, it follows that $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$ for all $\tilde{A} \in M_{\mathit{KB}}$.
- b) By construction of KB', $S_{\mathit{KB}'}[\iota/x]$ contains a constraint $x : \bigsqcup_{\tilde{A}} D_{\tilde{A}}$, and for one of the disjuncts $D_{\tilde{A}}$ its completion S' contains a constraint of the form $x : \{a_1, \ldots, a_n\} \sqcap \neg\tilde{A}$ with $a_i : \tilde{A} \in S$ for $i = 1 \ldots n$. Since S' is a completion to which none of the tableaux rules apply, the $\longrightarrow_{\sqcap}$- and the $\longrightarrow_{\mathcal{O}}$-rule have produced the constraints $a : \{a_1, \ldots, a_n\}$ and $a : \neg\tilde{A}$ in S' in which the variable x has been replaced by an individual a. As a solution for S', $(\mathcal{J}, \alpha^{\mathcal{J}})$ satisfies these two constraints and we have that both $\alpha^{\mathcal{J}}(a) \in (\Delta^{\mathcal{J}} \setminus \tilde{A}^{\mathcal{J}})$ and $\alpha^{\mathcal{J}}(a) \in \{\alpha^{\mathcal{J}}(a) \mid a : \tilde{A} \in S\}$ hold. This implies that $\alpha^{\mathcal{J}}(a) \notin \tilde{A}^{\mathcal{J}}$ and, since $(\mathcal{I}, \alpha^{\mathcal{I}})$ satisfies the constraint $a : \tilde{A}$, that $\alpha^{\mathcal{J}}(x) = \alpha^{\mathcal{I}}(a) \in \tilde{A}^{\mathcal{I}}$. From the arguments under b) we already know that $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$, and since we have an element $a^{\mathcal{I}}$ which is in $\tilde{A}^{\mathcal{I}}$ but not in $\tilde{A}^{\mathcal{J}}$, it follows that $\tilde{A}^{\mathcal{J}} \subset \tilde{A}^{\mathcal{I}}$.

$\Leftarrow$: Let S contain no clash. From Proposition 4 we know that there is a solution $(\mathcal{I}, \alpha^{\mathcal{I}})$ for S. We show by contradiction that there is no model $\mathcal{J}$ of KB such that $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$.

Assume that there is a model $\mathcal{J}$ of KB with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. First we show that for some $\mathcal{J}$-assignment $\alpha^{\mathcal{J}}$, $(\mathcal{J}, \alpha^{\mathcal{J}})$ is a solution for $S_{\mathit{KB}'}[\iota/x]$, where the knowledge base KB' is constructed according to Algorithm 1. Due to $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ we know that $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and $a^{\mathcal{J}} = a^{\mathcal{I}}$ for all individuals $a \in \Delta^{\mathcal{I}}$, and that for some $\tilde{A} \in M_{\mathit{KB}}$ there is an element $\iota^{\mathcal{J}} \in \Delta^{\mathcal{J}}$ which is in $\tilde{A}^{\mathcal{I}}$ but not in $\tilde{A}^{\mathcal{J}}$. Let $\alpha^{\mathcal{J}}_{x,\iota}$ be a $\mathcal{J}$-assignment with $\alpha^{\mathcal{J}}_{x,\iota}(x) = \iota^{\mathcal{J}}$. Since $\mathcal{J}$ is a model of KB, $(\mathcal{J}, \alpha^{\mathcal{J}}_{x,\iota})$ is a solution for S_{KB} due to Lemma 2. Moreover, as the individual ι is new to KB' and $\mathit{KB} \subset \mathit{KB}'$ by construction of KB', the replacement of ι by x does not affect any constraint in S_{KB} and we have that $S_{\mathit{KB}} \subset S_{\mathit{KB}'}[\iota/x]$. Hence, it suffices to show that the constraints in $S_{\mathit{KB}'}[\iota/x] \setminus S_{\mathit{KB}}$ are satisfied by $(\mathcal{J}, \alpha^{\mathcal{J}}_{x,\iota})$. For this purpose, we consider the axioms in $\mathit{KB}' \setminus \mathit{KB}$ that are inserted into KB' by Algorithm 1, and that can be a) concept inclusion axioms of the form $\tilde{A} \sqsubseteq \{a_1, \ldots, a_n\}$, or b) the concept assertion axiom $(\bigsqcup_{\tilde{A}} D_{\tilde{A}})(\iota)$ with disjuncts $D_{\tilde{A}}$ of the form $\neg\tilde{A} \sqcap \{a_1, \ldots, a_n\}$, for individuals $\{a_i \mid a_i : \tilde{A} \in S\}$ with $i \in \{1, \ldots, n\}$.

- a) For every $\tilde{A} \in M_{\mathit{KB}}$, KB' contains an axiom $\tilde{A} \sqsubseteq \{a_1, \ldots, a_n\}$ with individuals a_i that occur in concept constraints of the form $a_i : \tilde{A}$ within S. Since S is a completion, in any constraint of the form $x : \tilde{A}$ the variable x has been replaced by an individual $a \in \mathsf{N}_I^S$ in S due to the $\longrightarrow_{<_{\mathsf{CP}}}$-rule, such that for any constraint $o : \tilde{A} \in S$ we have that $o = a_i$ for some $i \in \{1, \ldots, n\}$. Since $\mathcal{I}$ is a solution for S, we have that $\tilde{A}^{\mathcal{I}} \subseteq \{\alpha^{\mathcal{I}}(a_1), \ldots, \alpha^{\mathcal{I}}(a_n)\} = \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}$. Since $\tilde{A}^{\mathcal{J}} \subseteq \tilde{A}^{\mathcal{I}}$ holds by assumption, $\mathcal{J}$ satisfies $\tilde{A}^{\mathcal{J}} \subseteq \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}$, and thus, the axiom $\tilde{A} \sqsubseteq \{a_1, \ldots, a_n\}$ for every $\tilde{A} \in M_{\mathit{KB}}$. If there are no assertions $\tilde{A}(a_i)$ in KB' then $\tilde{A}^{\mathcal{I}} = \emptyset$ and the respective axiom has the form $\tilde{A} \sqsubseteq \bot$. Hence, $(\mathcal{J}, \alpha^{\mathcal{J}}_{x,\iota})$ satisfies all the constraints $\forall x.x : C$ that result from these inclusion axioms in $S_{\mathit{KB}'}[\iota/x]$.
- b) Furthermore, due to the concept assertion $(\bigsqcup_{\tilde{A}} D_{\tilde{A}})(\iota)$ in KB', $S_{\mathit{KB}'}[\iota/x]$ contains the constraint $x : \bigsqcup_{\tilde{A}} D_{\tilde{A}}$ with disjuncts $D_{\tilde{A}}$ of the form $\neg\tilde{A} \sqcap \{a_1, \ldots, a_n\}$. Since from b) we know that $\tilde{A}^{\mathcal{I}} \subseteq$

$\{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}$, and since $a^{\mathcal{I}} = a^{\mathcal{J}}$ for all individuals a, we get that $\tilde{A}^{\mathcal{I}} = \{a_1^{\mathcal{J}}, \ldots, a_n^{\mathcal{J}}\} \subseteq \Delta^{\mathcal{J}}$. As for some $\tilde{A} \in M_{KB}$ the element $\iota^{\mathcal{J}}$ is in $\tilde{A}^{\mathcal{I}}$ but not in $\tilde{A}^{\mathcal{J}}$, we have that $\alpha_{x,\iota}^{\mathcal{J}}(x) \in \tilde{A}^{\mathcal{I}} \setminus \tilde{A}^{\mathcal{J}}$, and thus, $\alpha_{x,\iota}^{\mathcal{J}}(x) \in (\{a_1^{\mathcal{J}}, \ldots, a_k^{\mathcal{J}}\} \setminus \tilde{A}^{\mathcal{J}}) = (\Delta^{\mathcal{J}} \setminus \tilde{A}^{\mathcal{J}}) \cap \{\alpha_{x,\iota}^{\mathcal{J}}(a_1), \ldots, \alpha_{x,\iota}^{\mathcal{J}}(a_k)\}$. Hence, the pair $(\mathcal{J}, \alpha_{x,\iota}^{\mathcal{J}})$ satisfies the constraint $x : \bigsqcup_{\tilde{A}} D_{\tilde{A}}$ for some $\tilde{A} \in M_{KB}$ with $\tilde{A}^{\mathcal{J}} \subset \tilde{A}^{\mathcal{I}}$, as one of its disjuncts is satisfied.

Having shown that $(\mathcal{J}, \alpha_{x,\iota}^{\mathcal{J}})$ is a solution for $S_{KB'}[\iota/x]$, from Proposition 3($\Rightarrow$) and from Proposition 4($\Rightarrow$) it follows that there is a clash-free completion of $S_{KB'}[\iota/x]$. Hence, S must contain a preference clash, which contradicts the existence of $\mathcal{J}$. ◻

3.4 Sound and Complete Reasoning in Circumscriptive $\mathcal{ALCO}$

Finally, we can show that the presented calculus provides an effective procedure for reasoning with circumscribed knowledge bases, by the following theorems.

Theorem 1 (soundness). *Let KB be an $\mathcal{ALCO}$ knowledge base, CP a circumscription pattern and C an $\mathcal{ALCO}$ concept. If the repeated application of the tableaux rules results in a clash-free completion of the constraint system $S_{KB} \cup \{x : C\}$ then C is satisfiable with respect to $circ_{\mathsf{CP}}(KB)$.*

Proof. Let S be the clash-free completion of $S_{KB} \cup \{x : C\}$ produced by the tableaux procedure. Due to Proposition 5($\Leftarrow$), S is CP-solvable with respect to KB, and due to Proposition 3($\Leftarrow$), $S_{KB} \cup \{x : C\}$ also is. Finally, due to Proposition 2($\Leftarrow$), C is satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$. ◻

Theorem 2 (completeness). *Let KB be an $\mathcal{ALCO}$ knowledge base, CP a circumscription pattern and C an $\mathcal{ALCO}$ concept. If C is satisfiable with respect to $circ_{\mathsf{CP}}(KB)$ then some repeated application of the tableaux rules results in a clash-free completion of the constraint system $S_{KB} \cup \{x : C\}$.*

Proof. Let C be satisfiable with respect to $\mathrm{circ}_{\mathsf{CP}}(KB)$. Due to Proposition 2($\Rightarrow$), the constraint system $S_{KB} \cup \{x : C\}$ is CP-solvable with respect to KB, and due to Lemma 3 the tableaux procedure results in a completion S' of $S_{KB} \cup \{x : C\}$. From Proposition 3($\Rightarrow$), we know that S' is also CP-solvable with respect to KB. Finally, due to Proposition 5($\Rightarrow$), S' does not contain any clash. ◻

By Theorem 1 and Theorem 2, the proposed tableaux calculus is a sound and complete decision procedure for $\mathcal{ALCO}$ with concept circumscription.

4 Conclusion

We have presented a tableaux calculus for concept satisfiability with respect to circumscribed DL knowledge bases in the logic $\mathcal{ALCO}$. Building on tableaux procedures for classical DLs, the calculus checks a constraint system not only for clashes due to inconsistent concept assertion and individual naming, but also for preference clashes, which occur whenever the model associated with the constraint system is not minimal with respect to the preference relation $<_{\mathsf{CP}}$. This check is performed by testing a specifically constructed classical $\mathcal{ALCO}$ knowledge base for satisfiability, which requires reasoning in classical DL with nominals and equality between individuals.

We have proved that the presented calculus is a decision procedure for concept satisfiability in circumscriptive $\mathcal{ALCO}$, to which other reasoning tasks can be reduced. By this we have devised a first practical algorithmisation for description logic with circumscription that integrates well with state of the art tableaux methods for DL reasoning. This lays a basis for further investigations on optimisation of the calculus within the framework of tableaux procedures as a guided way for model construction. We have implemented a first prototype[7] of the calculus in Java that works together with ontology development tools, such as Protégé[8], via the DIG[9] interface. Notice also that our algorithm does not behave worse than the theoretical worst-case complexity, which was shown to be $\mathrm{NExp}^{\mathrm{NP}}$-complete in [4]. In our case, the call to the oracle corresponds to the calls to an $\mathcal{ALCO}$-reasoner for checking

[7] Available at `http://www.fzi.de/downloads/wim/sgr/CircDL.zip` .
[8] `http://protege.stanford.edu/`
[9] `http://dl.kr.org/dig/interface.html`

the preference clash, which is Exp-complete and thus as good as possible, since Exp is not known to be different from NP.

As future work we see the update of the calculus to support more expressive features, such as prioritisation between minimised concepts or the remaining constructs of the Web Ontology Language OWL [13] and its latest version OWL 1.1.[10] Moreover, optimisation issues need to be addressed to obtain a more efficient reasoning procedure. First ideas for specific optimisations would be to employ model caching techniques for the inner classical tableaux step as KB' might be identical in multiple cases, to postpone assertions of individuals to minimised predicates to avoid constructing non-minimal models, and to exploit early closing of tableaux branches through preference clash detection. Besides these, it would be interesting to see how well preferential tableaux performs when included in optimised state-of-the-art DL reasoners. Furthermore, as an open line for research it remains to develop a methodology for the formulation of appropriate circumscription patterns in various cases of non-monotonic reasoning, as pointed out in [4].

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[10] `http://owl1_1.cs.manchester.ac.uk/`