Abstract. Non-monotonic extensions of description logics (DLs) allow for default and local closed-world reasoning and are an acknowledged desired feature for applications, e.g. in the Semantic Web. A recent approach to such an extension is based on McCarthy’s circumscription, which rests on the principle of minimising the extension of selected predicates to locally close off dedicated parts of a domain model. While decidability and complexity results have been established in the literature, no practical algorithmisation for circumscriptive DLs has been proposed so far. In this paper, we present a tableaux calculus that can be used as a sound and complete decision procedure for concept satisfiability with respect to concept-circumscribed \textit{ALCO} knowledge bases. The calculus builds on existing tableaux for classical DLs, extended by the notion of a preference clash to detect the non-minimality of constructed models.

1 Introduction

Modern description logics (DLs) are formalisations of semantic networks and frame-based knowledge representation systems that build on classical logic. However, to also capture non-classical features, such as default and local closed-world reasoning, non-monotonic extensions to DLs have been investigated. While in the past such extensions were primarily devised on top of autoepistemic operators \cite{7} and default inclusions \cite{2}, a recent proposal \cite{4} is to extend DLs by circumscription and to perform non-monotonic reasoning on circumscribed DL knowledge bases. In circumscription, the extension of selected predicates – i.e. concepts or roles in the DL case – can be explicitly minimised to locally close off dedicated parts of a domain model, which results in a default reasoning behaviour. In contrast to the former approaches, non-monotonic reasoning in circumscriptive DLs also applies to “unknown individuals” not explicitly mentioned in a knowledge base, but whose existence is guaranteed due to existential quantification.

The proposal in \cite{4} presents a semantics for circumscriptive DLs together with decidability and complexity results, in particular for fragments of the logic \textit{ALCQIO}. However, a practical algorithmisation for reasoning in circumscriptive DLs has not been addressed so far. In this paper, we present such an algorithmisation that builds on existing DL tableaux methods. In particular, we present a tableaux calculus that supports reasoning with concept-circumscribed knowledge bases in the logic \textit{ALCO}. We focus on the reasoning task of concept satisfiability, which is motivated by an application of non-monotonic reasoning in a Semantic Web setting, described in \cite{9}. While typical examples in the circumscription literature deal with defeasible conclusions of circumscriptive abnormality theories, in this setting we use minimisation of concepts to realise a local closed-world assumption for the matchmaking of semantically annotated resources.

The reason for our choice of \textit{ALCO} as the underlying DL is twofold. First, we want to present the circumscriptive extensions for the simplest expressive DL \textit{ALC} for sake of a clear and concise description of the tableaux modifications. Second, there is the necessity to deal with nominals within the calculus in order to keep track of extensions of minimised concepts, so we include \textit{O}.

The basic idea behind our calculus is to detect the non-minimality of candidate models, produced by a tableaux procedure for classical DLs, via the notion of a preference clash, and based on the construction of a classical DL knowledge base that has a model if and only if the original candidate model produced is not minimal. This check can be realised by reasoning in classical DLs with nominals and equality between individuals. We formally prove this calculus to be a sound and complete decision procedure for concept satisfiability in circumscriptive \textit{ALCO}. A similar idea has been applied in \cite{12} for circumscriptive reasoning in first-order logic, where a tableaux for first-order formulas in clausal
Description Logics (DLs) [1] are typically fragments of first-order predicate logic that provide a well-studied formalism for knowledge representation systems. Circumscription [11], on the other hand, is an approach to non-monotonic reasoning based on the explicit minimisation of selected predicates. In this section, we present the description logic $\text{ALCO}$ extended with circumscription according to [4], which allows for non-monotonic reasoning with DL knowledge bases.

### 2.1 Circumscriptive $\text{ALCO}$

The basic elements to represent knowledge in DLs are individuals that represent concrete objects within a domain of discourse, concepts that group together individuals with common properties, and roles that put individuals in relation. The countably infinite sets $\mathbb{N}_I$, $\mathbb{N}_C$ and $\mathbb{N}_r$ of individual names, concept names and role names, respectively, form the basis to construct the syntactic elements of $\text{ALCO}$ according to the following grammar, in which $A \in \mathbb{N}_C$ denotes an atomic concept, $C, C_i$ denote complex concepts, $r \in \mathbb{N}_r$ denotes a role and $a_i \in \mathbb{N}_I$ denote individuals.

$$C, D \rightarrow \bot \mid \top \mid A \mid \neg C \mid C \cap D \mid C \cup D \mid \exists r . C \mid \forall r . C \mid \{a_1, \ldots, a_n\}$$

The semantics of the syntactic elements of $\text{ALCO}$ is defined in terms of an interpretation $I = (\Delta^I, \mathcal{I})$ with a non-empty set $\Delta^I$ as the domain and an interpretation function $\mathcal{I}$ that maps each individual $a \in \mathbb{N}_I$ to a distinct element $a^I \in \Delta^I$ and that interprets (possibly) complex concepts and roles as follows.

$$\begin{align*}
\top^I &= \Delta^I, \\
\bot^I &= \emptyset \\
A^I &= \Delta^I, \\
\neg C^I &= \Delta^I \setminus C^I \\
(C_1 \cap C_2)^I &= C_1^I \cap C_2^I \\
(C_1 \cup C_2)^I &= C_1^I \cup C_2^I \\
(-C)^I &= \Delta^I \setminus C^I \\
(\forall r . C)^I &= \{x \in \Delta^I \mid \forall y . (x, y) \in r^I \rightarrow y \in C^I\} \\
(\exists r . C)^I &= \{x \in \Delta^I \mid \exists y . (x, y) \in r^I \land y \in C^I\} \\
\{a_1, \ldots, a_n\}^I &= \{a_1^I, \ldots, a_n^I\}
\end{align*}$$

Notice that we assume unique names for individuals, i.e. $a_1^I \neq a_2^I$ for any interpretation $I$ and any pair $a_1, a_2 \in \mathbb{N}_I$.

An $\text{ALCO}$ knowledge base $KB$ is a set of axioms that are formed by concepts, roles and individuals. A concept assertion is an axiom of the form $C(a)$ that assigns the membership of an individual $a$ to a concept $C$. A role assertion is an axiom of the form $r(a_1, a_2)$ that assigns a directed relation between two individuals $a_1, a_2$ by the role $r$. A concept inclusion is an axiom of the form $C_1 \sqsubseteq C_2$ that states the subsumption of the concept $C_1$ by the concept $C_2$, while a concept equivalence axiom $C_1 \equiv C_2$ is a shortcut for two inclusions $C_1 \sqsubseteq C_2$ and $C_2 \sqsubseteq C_1$. An interpretation $I$ satisfies a concept assertion $C(a)$ if $a^I \in C^I$, a role assertion $r(a_1, a_2)$ if $(a_1^I, a_2^I) \in r^I$, a concept inclusion $C_1 \sqsubseteq C_2$ if $C_1^I \subseteq C_2^I$ and a concept equivalence $C_1 \equiv C_2$ if $C_1^I = C_2^I$. An interpretation that satisfies all axioms of a knowledge base $KB$ is called a model of $KB$. A concept $C$ is called satisfiable with respect to $KB$ if $KB$ has a model in which $C^I \neq \emptyset$ holds.

We now turn to the circumscription part of the formalism, that allows for non-monotonic reasoning by explicit minimisation of selected $\text{ALCO}$ concepts. We adopt a slightly simplified form of the circumscriptive DLs presented in [4] by restricting our formalism to parallel concept circumscription (without prioritisation among minimised concepts). For this purpose we define the notion of a circumscription pattern as follows.
Definition 1 (circumscription pattern, $\prec_{CP}$). A circumscription pattern\(^3\) $\mathbf{cp}$ is a tuple $(M, F, V)$ of sets of atomic concepts called the minimised, fixed and varying concepts. Based on $\mathbf{cp}$, a preference relation on interpretations is defined by setting $\mathcal{J} \prec_{CP} \mathcal{I}$ if and only if the following conditions hold:

(i) $\Delta^\mathcal{I} = \Delta^\mathcal{J}$ and $a^\mathcal{I} = a^\mathcal{J}$ for all $a^\mathcal{J} \in \Delta^\mathcal{J}$
(ii) $A^\mathcal{I} = A^\mathcal{J}$ for all $A \in F$
(iii) $\bar{A}^\mathcal{J} \subseteq \bar{A}^\mathcal{I}$ for all $\bar{A} \in M$
(iv) there is an $\bar{A} \in M$ such that $\bar{A}^\mathcal{J} \subset\bar{A}^\mathcal{I}$

For non-monotonic reasoning, a classical $\mathcal{ALCO}$ knowledge base is circumscribed with a circumscription pattern and reasoning is performed by means of the resulting circumscribed knowledge base, defined as follows.

Definition 2 (circumscribed knowledge base). A circumscribed knowledge base $\mathbf{circ}_{CP}(KB)$ is a knowledge base $KB$ together with a circumscription pattern $\mathbf{cp} = (M, F, V)$, such that the sets $M$, $F$ and $V$ partition the atomic concepts that occur in $KB$. An interpretation $\mathcal{I}$ is a model of $\mathbf{circ}_{CP}(KB)$ if $\mathcal{I}$ is a model of $KB$ and there exists no model $\mathcal{J}$ of $KB$ with $\mathcal{J} \prec_{CP} \mathcal{I}$.

The intuition behind the preference relation is to identify interpretations that are “smaller” in the extensions of minimised concepts than others, to select only the “smallest” ones as the preferred models of a knowledge base. Fixed concepts can be used to restrict this selection and to prevent certain models from being preferred.

2.2 Reasoning with Circumscribed Knowledge Bases

The typical DL reasoning tasks are defined as expected (see [4]) with respect to the models of a circumscribed knowledge base $\mathbf{circ}_{CP}(KB)$, which are just the preferred models of $KB$ with respect to $\mathbf{cp}$. For our calculus, we focus on concept satisfiability, which we define next. Other reasoning tasks can be reduced to concept satisfiability, as described in [4].

Definition 3 (circumscriptive concept satisfiability). A concept $C$ is satisfiable with respect to a circumscribed knowledge base $\mathbf{circ}_{CP}(KB)$ if some model $\mathcal{I}$ of $\mathbf{circ}_{CP}(KB)$ satisfies $C^\mathcal{I} \neq \emptyset$.

Observe that in classical DLs an atomic concept $A$ is satisfiable with respect to a knowledge base $KB$ “by default” if there is no evidence for its unsatisfiability in $KB$, i.e. any $A$ is satisfiable with respect to the empty knowledge base. Now suppose that $A$ is a minimised concept in a circumscription pattern $\mathbf{cp}$ by which $KB$ is circumscribed. Then, $A$ is unsatisfiable with respect to $\mathbf{circ}_{CP}(KB = \emptyset)$. Only if we explicitly assure that the extension of $A$ is non-empty, e.g. by setting $KB = \{A(a)\}$, $A$ becomes satisfiable.

A known result in circumscription is that there is a close relation between fixed and minimised predicates. Namely, fixed concepts are simulated by minimising them together with their complements, which is achieved through introducing additional concept names and respective equivalence axioms. The proofs of Lemma 1 and Proposition 1 are similar to that of [6, Theorem 1] but have been adapted from the first-order case to the setting of circumscriptive description logics.

Lemma 1 (coincidence of complementarily minimised concepts). Let $\bar{A}, \bar{B}$ be atomic concepts and let $KB$ be a knowledge base with $KB \models A \equiv \neg B$. Furthermore, let $\mathbf{cp} = (M, F, V)$ be a circumscription pattern with $\bar{A}, \bar{B} \in M$. For any two models $\mathcal{I}, \mathcal{J}$ of $KB$, $\mathcal{J} \prec_{CP} \mathcal{I}$ implies both $\bar{A}^\mathcal{J} \subseteq \bar{A}^\mathcal{I}$ and $\bar{B}^\mathcal{J} \subseteq \bar{B}^\mathcal{I}$.

Proof. Let $\mathcal{I}, \mathcal{J}$ be models of $KB$ with $\mathcal{J} \prec_{CP} \mathcal{I}$. According to condition (iii) from Definition 1, this implies a) $\bar{A}^\mathcal{J} \subseteq \bar{A}^\mathcal{I}$ and b) $\bar{B}^\mathcal{J} \subseteq \bar{B}^\mathcal{I}$, as both $\bar{A}$ and $\bar{B}$ are minimised. Since $\mathcal{I}$ and $\mathcal{J}$ are models of $KB$, the extension of $\bar{A}$ is just the complement of the extension of $\bar{B}$ in both $\mathcal{I}$ and $\mathcal{J}$. Hence, a) implies $\bar{B}^\mathcal{J} \supseteq \bar{B}^\mathcal{I}$ and b) implies $\bar{A}^\mathcal{J} \supseteq \bar{A}^\mathcal{I}$, such that we have both $\bar{A}^\mathcal{I} = \bar{A}^\mathcal{J}$ and $\bar{B}^\mathcal{I} = \bar{B}^\mathcal{J}$.

\(^3\)The notion of circumscription pattern introduced in [4] is more general and allows the sets $M$, $F$ and $V$ to also contain roles. There, a circumscription pattern according to Definition 1 is called a concept circumscription pattern. However, in the general case role circumscription leads to undecidability, which was also shown in [4]. As the calculus presented in this report does not allow for role circumscription, we use the term circumscription pattern to denote a concept circumscription pattern in the sense of [4].
Proposition 1 (simulation of concept fixation). Let $C$ be a concept, let $KB$ be a knowledge base and let $cp = (M, F, V)$ be a circumscription pattern with $F = \{ \hat{A}_1, \ldots, \hat{A}_n \}$. Furthermore, let
\[
KB' = KB \cup \{ \hat{A}_i \equiv \neg A_i \mid 1 \leq i \leq n \}
\]
and let
\[
cp' = (M \cup \{ \hat{A}_1, \ldots, \hat{A}_n, \hat{A}_1, \ldots, \hat{A}_n \}, \emptyset, V) ,
\]
where $\hat{A}_1, \ldots, \hat{A}_n$ are atomic concepts that do not occur in $KB$, $cp$ or $C$. Then, $C$ is satisfiable with respect to $\text{circ}_{cp}(KB)$ if and only if it is satisfiable with respect to $\text{circ}_{cp}(KB')$.

Proof. Assume that $C$ is satisfiable with respect to $\text{circ}_{cp}(KB)$ and let $I$ be a model of $\text{circ}_{cp}(KB)$ with $C^I \neq \emptyset$. Let $I'$ be an interpretation that coincides with $I$ except that $\hat{A}_i^I = A_i^I \setminus \hat{A}_i^I$ for $i = 1, \ldots, n$. Since none of the $\hat{A}_i$ occurs in $KB$, $I'$ is a model of $KB'$, and since $C^I = C^{I'} \neq \emptyset$, $C$ is satisfiable with respect to $KB'$. To prove that $C$ is also satisfiable with respect to $\text{circ}_{cp}(KB')$, we show by contradiction that $I'$ is minimal with respect to $<_{cp'}$.

Assume that $I'$ is not minimal with respect to $<_{cp'}$, such that there is a model $J'$ of $\text{circ}_{cp'}(KB')$ with $J' <_{cp'} I'$. By repeated application of Lemma 1, we know that $\hat{A}_i^{J'} = A_i^{J'}$ and $\hat{A}_i^{J''} = A_i^I$ hold for $i = 1, \ldots, n$. Hence, the minimised predicate $\hat{A}$ for that $\hat{A}_i^{J'} \subset A_i^I$ holds according to condition (iv) of Definition 1 must be in the set $M$ of the original circumscription pattern $cp$. Let $J$ be the interpretation that coincides with $J'$ except that $\hat{A}_i^J = \emptyset$ for $i = 1, \ldots, n$. Obviously, $J$ is a model of $KB$. We show that $J <_{cp} I$ holds by verifying the conditions (i)-(iv) from Definition 1. Condition (i) is trivially satisfied. From $\hat{A}_i^{J'} = \hat{A}_i^{J''}$ it follows that $\hat{A}_i^J = A_i^I$ for $i = 1, \ldots, n$, and hence condition (ii) is satisfied. Since $\hat{A}_i^{J''} \subseteq A_i^I$ holds for all $A \in M$, condition (iii) is satisfied. Since $\hat{A}_i^{J'} \subseteq A_i^I$ holds for some $A \in M$, also condition (iv) is satisfied. Thus, we have that $J <_{cp} I$ and $I$ cannot be minimal with respect to $cp$, which contradicts the existence of $J'$.

For the converse, assume that $C$ is satisfiable with respect to $\text{circ}_{cp'}(KB')$ and let $I'$ be a model of $\text{circ}_{cp'}(KB')$ with $C^{I'} \neq \emptyset$. Let $I$ be an interpretation that coincides with $I'$ except that $\hat{A}_i^I = \emptyset$ for $i = 1, \ldots, n$. Since none of the $\hat{A}_i$ appears in either $KB$ or $C$, $I$ is a model of $KB$ and $C^I \neq \emptyset$, such that $C$ is satisfied in $I$. To prove that $C$ is satisfiable with respect to $\text{circ}_{cp}(KB)$, we show by contradiction that $I$ is minimal with respect to $<_{cp}$.

Assume that $I$ is not minimal with respect to $<_{cp}$, such that there is a model $J$ of $\text{circ}_{cp}(KB)$ with $J <_{cp} I$. Let $I'$ be the interpretation that coincides with $J$ except that $\hat{A}_i^J = \emptyset$ for $i = 1, \ldots, n$. Obviously, $J$ is a model of $KB$. We show that $J' <_{cp} I'$ holds by verifying the conditions (i)-(iv) from Definition 1. Conditions (i) and (ii) are trivially satisfied. From $\hat{A}_i^{J'} = \hat{A}_i^{J''}$ we know that $\hat{A}_i^{I'} = \hat{A}_i^{I''}$ and also that $\hat{A}_i^{J'} = \hat{A}_i^{J''}$ for $i = 1, \ldots, n$, as any two $\hat{A}_i$, $\hat{A}_i$ have complementary extensions, and since $\hat{A}_i^{J''} \subseteq \hat{A}_i^I$ for all $A \in M$ due to $J <_{cp} I$, condition (iii) is also satisfied. From $\hat{A}_i^{J'} \subseteq \hat{A}_i^I$ for some $A \in M$ it follows that $\hat{A}_i^{J'} \subset \hat{A}_i^I$, satisfying condition (iv). Thus, we have that $J' <_{cp} I'$ and $I'$ cannot be minimal with respect to $cp'$, which contradicts the existence of $J'$.

To illustrate the reasoning task of checking concept satisfiability with respect to circumscribed knowledge bases we present the following example.

Example 1. The following knowledge base describes species living in the arctic sea.

\[
KB_1 = \{ \text{Bears(PolarBear)}, \neg \text{Bears(Blue Whale)}, \text{EndangeredSpecies(Blue Whale)} \}
\]

According to $KB_1$, the polar bear is a kind of bear, while the blue whale is not. Moreover, the blue whale is explicitly listed to be an endangered species, while the polar bear does not occur on this list. The following circumscription pattern allows to “switch off” the open-world assumption for the list of endangered species by minimising the extension of the concept EndangeredSpecies.

\[
\text{cp} = (M = \{ \text{EndangeredSpecies} \}, F = \emptyset, V = \{ \text{Bears} \})
\]

The concept Bears$\setminus$EndangeredSpecies is unsatisfiable with respect to the circumscribed knowledge base $\text{circ}_{cp}(KB_1)$, reflecting that there cannot be an individual that is both an endangered species and a kind of bear. The only endangered species in the preferred models of $KB_1$ is the blue whale, which is explicitly said to be no kind of bear.
Recently, however, the polar bear unfortunately had to be included in the list of endangered species, which is reflected by the following update of $KB_1$.

$$KB_2 = KB_1 \cup \{ \text{EndangeredSpecies(PolarBear)} \}$$

With respect to $\text{circcp}(KB_2)$, the concept $\text{Bears} \cap \text{EndangeredSpecies}$ is satisfiable, as the polar bear is a kind of bear and at the same time an endangered species in the preferred models of $KB_2$.

Instead of using a concept assertion for the explicitly mentioned individual $\text{PolarBear}$, we could alternatively update $KB_1$ by introducing an existentially quantified object through an inclusion axiom stating that the arctic sea is a habitat for an endangered bear species, as follows.

$$KB_3 = KB_1 \cup \{ \exists \text{isHabitatFor}.\text{Bears} \cap \text{EndangeredSpecies(ArcticSea)} \}$$

the concept $\text{Bears} \cap \text{EndangeredSpecies}$ is also satisfiable with respect to $\text{circcp}(KB_3)$. Observe that in any preferred model of $KB_3$ the extension of $\text{EndangeredSpecies}$ contains an unknown individual whose existence is propagated from the known individual $\text{ArcticSea}$ via the role $\text{isHabitatFor}$. Alternative approaches to non-monotonic reasoning in DL, such as [8, 2], typically treat unknown objects differently and do not allow for this kind of reasoning.

3 Tableaux Calculus for Circumscriptive $\mathcal{ALCO}$

In this section, we introduce a tableaux calculus that decides the satisfiability of a concept with respect to a circumscribed knowledge base. We build on the notion of constraint systems, which map to tableaux branches in tableaux calculi, and we keep the presentation similar to the related work in [5] and [7].

3.1 Constraint Systems and their Solvability

In addition to the alphabet of individuals $\mathbb{N}_I$, we introduce a set $\mathbb{N}_V$ of variable symbols. We denote elements of $\mathbb{N}_I$ by $a$, elements of $\mathbb{N}_V$ by $x$ and elements of $\mathbb{N}_I \cup \mathbb{N}_V$ by $o$, all possibly with an index.

A constraint is a syntactic entity of one of the forms

$$a : C, \quad (o_1, o_2) : r, \quad \forall x. x : C$$

where $C$ is an $\mathcal{ALCO}$ concept, $r$ is a role and the $o$’s are objects in $\mathbb{N}_I \cup \mathbb{N}_V$. A constraint system, denoted by $S$, is a finite set of constraints. Let $\mathbb{N}^S_I$ denote the individuals and let $\mathbb{N}^S_V$ be the variables that occur in a constraint system $S$, respectively.

Given an interpretation $I$, we define an $I$-assignment as a function $\alpha^I : \mathbb{N}_I \cup \mathbb{N}_V \mapsto \Delta^I$, that maps every variable of $\mathbb{N}_V$ to an element of $\Delta^I$, and every individual $a$ to $\alpha^I(a) = a^I$ for all $a \in \mathbb{N}_I$.

A pair $(I, \alpha^I)$ of an interpretation $I$ and an $I$-assignment $\alpha^I$ satisfies a constraint $o : C$ if $\alpha^I(o) \in C^I$, a constraint $(o_1, o_2) : r$ if $(\alpha^I(o_1), \alpha^I(o_2)) \in r^I$ and a constraint $\forall x. x : C$ if $C^I = \Delta^I$.

A solution for a constraint system $S$ is a pair $(I, \alpha^S)$ of an interpretation $I$ and an $I$-assignment $\alpha^S$ that satisfies all constraints in $S$.

We denote by $S[o_1/o_2]$ the constraint system that is obtained by replacing any occurrence of object $o_1$ by object $o_2$ in every constraint in $S$. Furthermore, we define the constraint system $S_{KB}$ to be obtained from an $\mathcal{ALCO}$ knowledge base $KB$ by including one constraint of the form $a : C$ for each concept assertion $C(a) \in KB$, one constraint $(a_1, a_2) : r$ for each role assertion $r(a_1, a_2) \in KB$ and one constraint $\forall x. x \in \lnot C_1 \cup C_2$ for each concept inclusion $C_1 \subseteq C_2 \in KB$, such that $S_{KB}$ captures all the information in $KB$. We assume all complex concepts that occur in $KB$ to be in negation normal form, as described in [15].

To ensure termination of our calculus in the presence of general inclusion axioms, we further need to introduce the notion of blocking (see e.g. [5]). We say that an object $o_1$ is a direct predecessor of an object $o_2$ if the respective constraint system $S$ contains a role constraint $(o_1, o_2) : r$ for some role $r$. We denote by predecessor the transitive closure of the direct predecessor relation. Moreover, we say that, in a constraint system $S$, an object $o_2$ is blocked by an object $o_1$ if $o_1$ is a predecessor of $o_2$ and if $\{ C : o_2 : C \in S \} \subseteq \{ C : o_1 : C \in S \}$ holds.

Due to the analogy between a constraint system and a knowledge base the following Lemma holds.
Lemma 2. Let $KB$ be an ALCO knowledge base, $S$ be a constraint system with $S_{KB} \subseteq S$ and $I$ be an interpretation. If $I$ is a model of $KB$ then, for any $I$-assignment $\alpha^I$, $(I, \alpha^I)$ is a solution for $S_{KB}$. Furthermore, for any solution $(I, \alpha^I)$ for $S$, $I$ is a model of $KB$.

Proof. $\Rightarrow$: Assume that $I$ is a model of $KB$. As no variables occur in $S_{KB}$, all its concept and role constraints are of the form $a:C$ and $(a_1, a_2):r$, respectively. Since $I$ is a model of $KB$, $a^I \in C^I$ holds for any concept constraint and $(a^I_1, a^I_2) \in r^I$ for any role constraint in $S_{KB}$. As for any assignment $\alpha^I(a) = a^I$, $(I, \alpha^I)$ satisfies every concept and role constraint in $S_{KB}$. Moreover, universal constraints in $S_{KB}$ have the form $\forall x. x : \neg C_1 \sqcup C_2$. Since $I$ satisfies their original concept inclusion $C_1 \subseteq C_2$, all universal constraints in $S_{KB}$ are satisfied as well due to the fact that $C_1^I \subseteq C_2^I$ implies $(\Delta^I \setminus C_1^I) \cup C_2^I = \Delta^I$. Hence, $(I, \alpha^I)$ satisfies all constraints in $S_{KB}$ and is therefore a solution for it.

$\Leftarrow$: Assume that $(I, \alpha^I)$ is a solution for $S$. Since $(I, \alpha^I)$ satisfies all constraints in $S$ and $S_{KB} \subseteq S$, it satisfies all constraints in $S_{KB}$ in particular. Hence, it holds that $\alpha^I(o) \in C^I$ for every concept constraint, and $(\alpha^I(o_1), \alpha^I(o_2)) \in r^I$ for every role constraint in $S_{KB}$. Moreover, $(\Delta^I \setminus C_1^I) \cup C_2^I = \Delta^I$ holds for every universal constraint $\forall x. x : \neg C_1 \sqcup C_2 \in S_{KB}$, which implies $C_1^I \subseteq C_2^I$. Since there is a one-to-one correspondence between constraints in $S_{KB}$ and axioms in $KB$, $I$ satisfies every concept assertion, role assertion and concept inclusion in $KB$ and is therefore a model of $KB$. $\square$

The calculus we present is based on finding a solution for constraint systems the interpretation of which is a preferred model of an initial knowledge base with respect to a circumscription pattern. For this purpose we define the notion of solvability.

Definition 4 (cp-solvability). A constraint system $S$ is cp-solvable with respect to $KB$ if there is a model $I$ of $KB$ and an $I$-assignment $\alpha^I$ such that $(I, \alpha^I)$ is a solution for $S$ and there is no model $J$ of $KB$ with $J <_{cp} I$.

By the next proposition, we reduce circumscriptive concept satisfiability to checking a constraint system for its solvability.

Proposition 2 (satisfiability reduction). Let $KB$ be an ALCO knowledge base, $cp$ be a circumscription pattern and $C$ be an ALCO concept. $C$ is satisfiable with respect to $\text{circ}_{cp}(KB)$ if and only if $S_{KB} \cup \{x : C\}$ is cp-solvable with respect to $KB$.

Proof.

$\Rightarrow$: Since $C$ is satisfiable with respect to $\text{circ}_{cp}(KB)$, there is a model $I$ of $\text{circ}_{cp}(KB)$ in which $C^I$ is nonempty. Let $a$ be an individual with $a^I \in C^I$. Since $I$ is also a model of $KB$ and due to Lemma 2, $(I, \alpha^I)$ is a solution for $S_{KB}$ for any $I$-assignment $\alpha^I$. Let $\alpha^I_{x,a}$ be an $I$-assignment with $\alpha^I_{x,a}(x) = a^I$. Then, $(I, \alpha^I_{x,a})$ satisfies, besides the constraints in $S_{KB}$, also the constraint $x : C$, because of $\alpha^I_{x,a}(x) \in C^I$, and is therefore a solution for $S_{KB} \cup \{x : C\}$. Since there is no other model $J$ of $KB$ with $J <_{cp} I$, $S_{KB} \cup \{x : C\}$ is cp-solvable with respect to $KB$.

$\Leftarrow$: Since $S_{KB} \cup \{x : C\}$ is cp-solvable with respect to $KB$, there is a model $I$ of $KB$ and an $I$-assignment $\alpha^I$ such that $(I, \alpha^I)$ is a solution for $S_{KB} \cup \{x : C\}$. Moreover, there exists an element $a^I \in C^I$ with $\alpha^I(x) = a^I \in C^I$ because $(I, C^I)$ satisfies the constraint $x : C$. By definition of cp-solvability, there is no model $J$ of $KB$ with $J <_{cp} I$, and thus, $I$ is a model of $\text{circ}_{cp}(KB)$ in which $C^I$ is non-empty. Hence, $C$ is satisfiable with respect to $\text{circ}_{cp}(KB)$. $\square$

3.2 Tableaux Expansion Rules

Constraint systems are manipulated by tableau expansion rules, which decompose the structure of complex logical constructs or replace variables by concrete individuals. By expanding a constraint system with the resulting constraints, our calculus tries to build a model for the initial knowledge base that is represented by the constraint system. To decide the satisfiability of a concept $C$ with respect to a circumscribed knowledge base $\text{circ}_{cp}(KB)$ according to Proposition 2, we initialise the calculus with the constraint system $S_{KB} \cup \{x : C\}$ and repeatedly apply the tableau rules given in Table 3.2.

We assume that fixed concepts are simulated by minimising them together with their complements according to Proposition 1, such that no fixed concepts occur in $cp$. 

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finite projection of the potentially infinite set ing constraint system. Contrarily, the rules \(\rightarrow_{\pi}\) and \(\rightarrow_{\tau}\) are deterministic and their application yields a single resulting constraint system. Contrarily, the rules \(\rightarrow_{\eta}\), \(\rightarrow_{\sigma}\) and \(\rightarrow_{<_{CP}}\) are non-deterministic, meaning that they can be applied in multiple ways that yield different constraint systems. Any such non-deterministic choice produces a branching point for backtracking when algorithmically determined. In the \(\rightarrow_{\eta}\)-rule, the disjunction leads to the choice of expanding on either of the disjuncts, while in the \(\rightarrow_{\sigma}\)- and the \(\rightarrow_{<_{CP}}\)-rule the presence of several individuals leads to the choice of selecting one for replacement of the variable \(x\). Moreover, the \(\rightarrow_{<_{CP}}\)-rule introduces new individuals into the constraint system whenever \(i\) is selected for replacement\(^4\), while the \(\rightarrow_{\eta}\)-rule introduces new variables whenever an object lacks a role filler.

**Definition 5 (completion).** A completion of a constraint system \(S\) with regard to \(cp\) and \(KB\) is any constraint system that results from the application of the tableaux rules to \(S\), using \(cp\) and \(KB\), and to which none of the rules is applicable. (Often the parameters \(cp\) and \(KB\) for rule application are clear from the context and are omitted.)

The repeated application of rules finally leads to a completion of the initial constraint system that contains the exhaustive decomposition of complex constraints, which is established by the following lemma.

**Lemma 3 (termination).** Let \(S\) be a constraint system. A repeated application of the tableaux rules to \(S\) ultimately terminates, and yields a completion of \(S\).

**Proof (Sketch).** After application of any of the rules, \(S\) is altered such that the rule condition is not triggered again with the same parameters. Complex concepts that occur in constraints are decomposed by the rules until their finite structure finally breaks down to atomic concepts, nominals and their negations. Due to the finite structure of complex concepts and the blocking condition (see e.g. [5]) employed in the \(\rightarrow_{\eta}\)-rule only finitely many variables are introduced, and whenever the \(\rightarrow_{\sigma}\)-rule or the \(\rightarrow_{<_{CP}}\)-rule is applied to a nominal or an atomic concept, a variable is replaced by an individual, such that the number of variables cannot grow infinitely. Hence, the resulting constraint system reaches a point at which no rule is applicable to it and is then, by definition, a completion of \(S\).

Moreover, we establish the result that the tableaux expansion rules of our calculus preserve the solvability of constraint systems as follows.

\(^4\) The idea of including a new individual \(i\) as a representative for the infinitely many remaining objects in \(N_I \setminus N_I^S\) in the domain is taken from [7].
Proposition 3 (solvability preservation). Let $KB$ be an $\mathcal{ALCO}$ knowledge base, $CP$ be a circumscription pattern and $S, S'$ be two constraint systems.

1. If $S'$ is obtained from $S$ by application of a deterministic rule then $S$ is $CP$-solvable with respect to $KB$ if and only if $S'$ is $CP$-solvable with respect to $KB$.

2. If $S'$ is obtained from $S$ by application of a non-deterministic rule then $S$ is $CP$-solvable with respect to $KB$ if $S'$ is $CP$-solvable with respect to $KB$. Furthermore, if $S$ is $CP$-solvable with respect to $KB$ and a non-deterministic rule applies to $S$ then it can be applied in such a way that the resulting constraint system $S'$ is also $CP$-solvable with respect to $KB$.

Proof. For the first part, we consider the deterministic rules:

$\Leftarrow$: Assume that $S'$ is obtained from $S$ by application of a deterministic rule and that $S'$ is $CP$-solvable with respect to $KB$. Let $(I, \alpha^I)$ be a solution for $S'$ such that $I$ is a model of $KB$ and there is no model $J$ of $KB$ with $J \prec CP I$. For any of the deterministic rules it holds that $S$ is a subset of $S'$, and therefore $(I, \alpha^I)$ is also a solution for $S$. Since there is no model $J$ of $KB$ with $J \prec CP I$, $S$ is $CP$-solvable with respect to $KB$.

$\Rightarrow$: Assume that $S'$ is obtained from $S$ by application of a deterministic rule and that $S$ is $CP$-solvable with respect to $KB$. Let $(I, \alpha^I)$ be a solution for $S$ such that $I$ is a model of $KB$ and there is no model $J$ of $KB$ with $J \prec CP I$. We subsequently consider the various deterministic rules.

$\to$-

If the $\to$-rule has been applied then $S$ contains a constraint of the form $\forall x. x : C$ and the resulting constraint system $S'$ contains the concept constraint $o : C$ for some $o \in N^I_S \cup N^I_D$. As a solution for $S$, $(I, \alpha^I)$ satisfies $\forall x. x : C$ and we have that $C^I = \Delta^I$. From $(N^I_S \cup N^I_D) \subseteq \Delta^I$ it follows that $\alpha^I(o) \in \Delta^I$, and hence this concept constraint is also satisfied by $(I, \alpha^I)$, which is therefore a solution for $S'$.

$\to$-

If the $\to$-rule has been applied then $S$ contains a constraint of the form $o : C \cap D$ and $S'$ contains $\{o : C, o : D\}$. As a solution for $S$, $(I, \alpha^I)$ satisfies the constraint $o : C \cap D$, and we have that $\alpha^I(o) \in (C \cap D)^I$ and, due to the semantics of $\mathcal{ALCO}$, both $\alpha^I(o) \in C^I$ and $\alpha^I(o) \in D^I$ hold. Hence, $(I, \alpha^I)$ also satisfies the two constraints $o : C$ and $o : D$, and is therefore a solution for $S'$.

$\to$-

If the $\to$-rule has been applied then $S$ contains a constraint of the form $o : \exists r. C$ and $S'$ contains the two constraints $(o, x) : r$ and $x : C$, where $x \notin N^I_S$ is a new variable. As a solution for $S$, $(I, \alpha^I)$ satisfies the constraint $o : \exists r. C$, and we have that $\alpha^I(o) \in (\exists r. C)^I$. Due to the semantics of $\mathcal{ALCO}$ there is some $a^I \in \Delta^I$ for which both $(\alpha^I(o), a^I) \in r^I$ and $a^I \in C^I$ hold. Since $x$ is new to $S$, we can safely assume that $\alpha^I(x) = a^I$. Then, $(I, \alpha^I)$ also satisfies the two constraints $(o, x) : r$ and $x : C$, and is therefore a solution for $S'$.

$\to$-

If the $\to$-rule has been applied then $S$ contains two constraints $o_1 : \forall r. C$ and $(o_1, o_2) : r$, while the resulting constraint system $S'$ differs by $S' \setminus S = \{o_2 : C\}$. Since $(I, \alpha^I)$ satisfies $S$, it also satisfies $o_1 : \forall r. C$ and $(o_1, o_2) : r$, and we have that $\alpha^I(o_1) \in (\forall r. C)^I$ and $(\alpha^I(o_1), \alpha^I(o_2)) \in r^I$. Due to the semantics of $\mathcal{ALCO}$, this implies that $\alpha^I(o_2) \in C^I$, and thus, $(I, \alpha^I)$ also satisfies $o_2 : C$ and is therefore a solution for $S'$.

Since in all of the cases $(I, \alpha^I)$ is a solution for $S'$ and there is no model $J$ of $KB$ with $J \prec CP I$ by assumption, $S'$ is $CP$-solvable with respect to $KB$.

For the second part, we consider the non-deterministic rules:

$\Leftarrow$: Assume that $S'$ is obtained from $S$ by application of a non-deterministic rule and that $S'$ is $CP$-solvable with respect to $KB$. Let $(I, \alpha^I)$ be a solution for $S'$ such that $I$ is a model of $KB$ and there is no model $J$ of $KB$ with $J \prec CP I$. We subsequently consider the various non-deterministic rules.

$\to$-

If the $\to$-rule has been applied then $S$ contains a constraint of the form $o : C \cup D$ and the resulting constraint system $S'$ differs from $S$ by either $S' \setminus S = \{o : C\}$ or $S' \setminus S = \{o : D\}$. Without loss of generality we can assume that $S' \setminus S = \{o : C\}$. Since $(I, \alpha^I)$ is a solution for $S'$, it also satisfies $o : C$, and we have that $\alpha^I(o) \in C^I$, and thus also $\alpha^I(o) \in C^I \cup D^I$. Hence, $(I, \alpha^I)$ satisfies $o : C \cup D$ and is therefore a solution for $S$. 

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W e distinguish the two cases in which a) a constraint system \( S \) is solvable with respect to \( K B \) and b) in case \( S \) is not solvable with respect to \( K B \). Let \( I \) be a solution for \( S \) such that \( I \) is a model of \( K B \) and there is no model \( J \) of \( KB \) with \( J <_{CP} I \) by assumption, \( S \) is \( CP \)-solvable with respect to \( K B \).

\[ \Rightarrow \text{: Assume that } S' \text{ is solvable from } S \text{ by application of a non-deterministic rule and } S' \text{ is CP-solvable with respect to } K B. \text{ Let } (I, \alpha^I) \text{ be a solution for } S \text{ such that } I \text{ is a model of } K B \text{ and there is no model } J \text{ of } KB \text{ with } J <_{CP} I. \text{ We subsequently consider the various non-deterministic rules.} \]

\[ \leftarrow \text{c}: \text{If the } \leftarrow \text{c-rule has been applied then } S \text{ contains a constraint of the form } o : C \sqcup D \text{ and the resulting constraint system } S' \text{ differs from } S \text{ by either } S' \setminus S = \{o : C\} \text{ or } S' \setminus S = \{o : D\}. \text{ Without loss of generality we can assume that } S' \setminus S = \{o : C\}, \text{ and that } \alpha^I(o) \in C^I \text{ holds for the solution } (I, \alpha^I) \text{ for } S. \text{ Then, } (I, \alpha^I) \text{ also satisfies } o : C, \text{ and is therefore a solution for } S'. \text{ Since there is no model } J \text{ of } KB \text{ with } J <_{CP} I \text{ by assumption, the } \leftarrow \text{u-rule can be applied to } S \text{ in such a way that } S' \text{ is CP-solvable with respect to } K B. \]

\[ \leftarrow \text{c}: \text{If the } \leftarrow \text{c-rule has been applied then } S \text{ contains a constraint of the form } x : \{a_1, \ldots, a_k\}. \text{ As a solution for } S, (I, \alpha^I) \text{ satisfies this constraint and there is some individual } a_i \in \{a_1, \ldots, a_k\} \text{ with } \alpha(x) = a_i^I \text{ that can be picked for the application of the } \leftarrow \text{c-rule to yield } S' = S[x/a_i]. \text{ Since the constraint } a_i : \{a_1, \ldots, a_k\} \text{ with } i \in \{1, \ldots, k\} \text{ is trivially satisfied, } (I, \alpha^I) \text{ is a solution for } S[x/a_i], \text{ and since there is no model } J \text{ of } KB \text{ with } J <_{CP} I \text{ by assumption, the } \leftarrow \text{c-rule can be applied to } S \text{ in such a way that } S' \text{ is CP-solvable with respect to } K B. \]

\[ \leftarrow \text{c}: \text{If the } \leftarrow \text{c-rule has been applied then } S \text{ contains a constraint of the form } x : A \text{ with } A \in M_{KB}. \text{ As a solution for } S, (I, \alpha^I) \text{ satisfies this constraint and there is some individual } a \in \mathbb{N}_I \text{ with } \alpha^I(x) = a. \text{ We distinguish the two cases in which a) } a \text{ is in } \mathbb{N}_I^S \text{ and b) } a \text{ is a new individual not in } \mathbb{N}_I^S:} \]

- a) In case \( a \in \mathbb{N}_I^S \), \( a \) can be picked for the application of the \( \leftarrow \text{c-rule and it directly follows that } (I, \alpha^I) \text{ is a solution for the resulting constraint system } S' = S[x/a]. \]

- b) In case \( a \in \mathbb{N}_I \setminus \mathbb{N}_I^S, i \in \mathbb{N}_I \setminus \mathbb{N}_I^S \) can be picked for the application of the \( \leftarrow \text{c-rule as a representative for any new individual. Then, } S[x/a] \text{ and } S[x/i] \text{ differ only by the naming of an individual new to } S \text{ and are in this sense isomorphic.}^5 \text{ Hence, as } (I, \alpha^I) \text{ is a solution for } S[x/a] \text{ it is also a solution for the resulting constraint system } S[x/i] = S'. \]

Finally, since in all of the cases \( (I, \alpha^I) \) is a solution for \( S' \) and there is no model \( J \) of \( KB \) with \( J <_{CP} I \) by assumption, the \( \leftarrow \text{c-rule can be applied to } S \text{ in such a way that } S' \text{ is CP-solvable with respect to } K B. \]

\[ \square \]

### 3.3 Notions of Clash and Detection of Inconsistencies

Once a completion of an initial constraint system has been produced, its solvability can be verified by using the notion of a clash. In addition to the clashes defined in [7, 14], which represent obvious contradictions in a knowledge base, we additionally introduce the notion of a preference clash, which reflects non-minimality of the respective model with regard to the preference relation \( <_{CP} \).

**Definition 6 (Clashes).** Let \( S \) be a constraint system.

\( S \) contains an inconsistency clash if at least one of the following holds:

(i) \( S \) contains a constraint of the form \( o : \bot. \)

(ii) \( S \) contains two constraints of the form \( o : A, \ o : \neg A. \)

\( S \) contains an individual clash if at least one of the following holds:

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5 See also the analogous argument in [7, Lemma 3.6].
S contains a constraint of the form \( a : \{a_1, \ldots, a_k \} \).
with \( a \neq a_i \) for all \( i \in \{1, \ldots, k \} \subset \mathbb{N} \).

S contains a constraint of the form \( \neg \{a_1, \ldots, a_k \} \).
with \( a = a_i \) for some \( i \in \{1, \ldots, k \} \subset \mathbb{N} \).

S contains a preference clash, parameterised with a circumscription pattern \( \mathcal{CP} \) and an \( \mathcal{ALCO} \) knowledge base \( \mathcal{KB} \), if the following condition holds:

(v) the constraint system \( S_{\mathcal{KB}}[ι/x] \) has a completion, with regard to
\( \mathcal{CP}' = (\emptyset, \emptyset, F \cup M \cup V) \) and \( \mathcal{KB}' \), that does neither contain an
inconsistency clash nor an individual clash, while the \( \mathcal{ALCO} \) knowledge base \( \mathcal{KB}' \) is constructed according to Algorithm 1.

Algorithm 1

\textbf{Construct a knowledge base} \( \mathcal{KB}' \).

\textbf{Require:} a constraint system \( S \) produced for an initial \( \mathcal{ALCO} \) knowledge base \( \mathcal{KB} \) circumscribed with a
circumscription pattern \( \mathcal{CP} = (M, F, V) \)

\[ D \leftarrow \{ \bot \} \]
\[ \mathcal{KB}' \leftarrow \mathcal{KB} \]

\textbf{for all} \( \tilde{A} \in M_{\mathcal{KB}} \) \do

\textbf{if} there are constraints \( a_1 : \tilde{A}, \ldots, a_n : \tilde{A} \in S \) \then
\[ \mathcal{KB}' \leftarrow \mathcal{KB}' \cup \{ \tilde{A} \sqsubseteq \{a_1, \ldots, a_n \} \} \]
\[ D \leftarrow D \cup \{ \{a_1, \ldots, a_n \} \cap \neg \tilde{A} \} \]
\textbf{else}
\[ \mathcal{KB}' \leftarrow \mathcal{KB}' \cup \{ \tilde{A} \sqsubseteq \bot \} \]
\textbf{end if}
\textbf{end for}

\[ \mathcal{KB}' \leftarrow \mathcal{KB}' \cup \{ \bigcup_{D_A \in D} D_A(ι) \} \], with \( ι \) a new individual

The idea behind the construction of \( \mathcal{KB}' \) in Algorithm 1 is to freeze the instance situation for
minimised concepts as asserted in the current constraint system perceived as reflecting some model \( \mathcal{I} \) of the original knowledge base \( \mathcal{KB} \). Then, \( \mathcal{KB}' \) is constructed such that for any of its models \( \mathcal{J} \) it
holds that \( \mathcal{J} <_{\mathcal{CP}} \mathcal{I} \), and thus, checking \( \mathcal{KB}' \) for unsatisfiability verifies minimality of \( \mathcal{I} \). By inclusion
axioms for minimised concepts \( \tilde{A} \), condition 3 of Definition 1 is assured to hold for each model of \( \mathcal{KB}' \).
Moreover, by the disjunctive concept assertion, condition 4 of Definition 1 is assured to hold, such
that any model of \( \mathcal{KB}' \) is actually “smaller” than \( \mathcal{I} \) in some minimised concept, which is achieved
by mapping the not uniquely named individual \( ι \) to one that already occurs in the extension of a
minimised concept. Although in general we assume unique names in the formalism, the replacement
of the new individual \( ι \) by the variable \( x \) within \( S_{\mathcal{KB}}[ι/x] \) in condition (v) of Definition 6 allows \( ι \) to be
(indirectly) identified with some other individual.

We illustrate the detection of clashes in our calculus by means of an example.

\textbf{Example 2.} Consider the circumscribed knowledge base \( \text{circ}_{\mathcal{CP}}(\mathcal{KB}) \) with the following knowledge base
\( \mathcal{KB} \) and circumscription pattern \( \mathcal{CP} \).

\[ \mathcal{KB} = \{ \neg \text{Bears} (\text{BlueWhale}) , \text{EndangeredSpecies} (\text{BlueWhale}) \} \]
\[ \mathcal{CP} = (M = \{ \text{EndangeredSpecies} \}, F = \emptyset, V = \{ \text{Bears} \}) \]

We perform our calculus to check whether the concept \( \text{Bears} \sqcap \text{EndangeredSpecies} \) is satisfiable with
respect to \( \text{circ}_{\mathcal{CP}}(\mathcal{KB}) \).

We start with the constraint system initialised as follows.

\[ S_{\mathcal{KB}} \cup \{ x : \text{Bears} \sqcap \text{EndangeredSpecies} \} = \{ \text{BlueWhale} : \neg \text{Bears} , \]
\[ \text{BlueWhale} : \text{EndangeredSpecies} , x : \text{Bears} \sqcap \text{EndangeredSpecies} \} \]
From the application of the \(\rightarrow_{\tau}\)-rule and the non-deterministic \(\rightarrow_{CP}\)-rule, the following two resulting completions are produced.

\[
S_1 = \{ \text{Blue Whale} : \neg\text{Bears}, \text{Blue Whale} : \text{Endangered Species}, \text{Blue Whale} : \text{Bears} \}
\]

\[
S_2 = \{ \text{Blue Whale} : \neg\text{Bears}, \text{Blue Whale} : \text{Endangered Species}, \text{t}_0 : \text{Bears}, \text{t}_0 : \text{Endangered Species} \}
\]

The completion \(S_1\) obviously contains an inconsistency clash, since it contains both the constraints \(\text{Blue Whale} : \text{Bears}\) and \(\text{Blue Whale} : \neg\text{Bears}\).

For the completion \(S_2\), we construct the knowledge base \(KB'\) according to Algorithm 1 as follows.

\[
KB' = \{ \neg\text{Bears(Blue Whale)} , \text{Endangered Species(Blue Whale)} , \text{Endangered Species} \subseteq \{\text{Blue Whale, t}_0\} , \neg\text{Endangered Species} \cap \{\text{Blue Whale, t}_0\}(t) \}
\]

To check whether the completion \(S_2\) contains a preference clash, we need to check if \(KB'\) has a model, while the new individual \(\iota\) is not uniquely named. In our calculus, this is done by trying to produce a clash-free completion of the following respective constraint system.

\[
S_{KB'}[\iota/x] = \{ \text{Blue Whale} : \neg\text{Bears}, \text{Blue Whale} : \text{Endangered Species}, \forall x. x : \neg\text{Endangered Species} \cap \{\text{Blue Whale, t}_0\} , \exists x : \neg\text{Endangered Species} \cap \{\text{Blue Whale, t}_0\} \}
\]

The constraint system \(S_{KB'}[\iota/x]\) has the following completion \(S'\), in which the new variable \(x\) has been replaced by the individual \(t_0\) that was introduced by the \(\rightarrow_{CP}\)-rule in the completion process for \(S_2\) before.

\[
S' = S_{KB'}[\iota/x] \cup \{ t_0 : \neg\text{Endangered Species}, t_0 : \{\text{Blue Whale, t}_0\} \}
\]

\(S'\) does neither contain an inconsistency clash nor an individual clash, and thus, \(S_2\) contains a preference clash according to condition (v) of Definition 6. Since both \(S_1\) and \(S_2\) contain some clash, the initial constraint system \(S_{KB} \cup \{ x : \neg\text{Endangered Species} \}\) has no clash-free completion. Hence, due to Proposition 2 the concept \(\text{Bears} \cap \text{Endangered Species}\) is unsatisfiable with respect to \(\circ_{CP}(KB)\).

In the description logic literature [5, 14, 3, 10], tableau methods for sound and complete reasoning have been proposed for various DL variants including \(\text{ALCO}\). They detect inconsistencies in DL knowledge bases by checking completions of constraint systems for the occurrence of a clash. For sake of completeness, we include this result in our presentation in form of the following proposition.

**Proposition 4 (ALCO correctness)**. Let \(KB\) be an \(\text{ALCO}\) knowledge base and \(S\) be the completion of a constraint system that contains at least the constraints of \(S_{KB}\), with regard to any circumscription pattern and \(KB\). Then \(S\) has a solution if and only if it contains neither an inconsistency clash nor an individual clash.

**Proof.**

\(\Rightarrow:\) Assume that \((I, \alpha^{\mathcal{I}})\) is a solution for \(S\), for some model \(I\) of \(KB\) and an \(I\)-assignment \(\alpha^{\mathcal{I}}\). We show by contradiction that \(S\) does not contain an inconsistency clash nor an individual clash.

\(\neg\text{inconsistency clash:}\)

Assume that \(S\) contains an inconsistency clash. Then, it contains either a constraint \(o : \bot\) or two constraints \(\{o : A, o : \neg A\}\), and \(I\) must satisfy \(\alpha^{\mathcal{I}}(o) \in A^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus A^{\mathcal{I}}) = \emptyset\). Hence, \(I\) cannot be a model of \(KB\), which contradicts \(S\) to contain an inconsistency clash.

\(\neg\text{individual clash:}\)

Assume that \(S\) contains an individual clash. Then, it contains either a constraint \(a : \{a_1, \ldots, a_n\}\) with \(a \neq a_i\) for all \(i \in \{1, \ldots, n\}\) or a constraint \(\neg \{a_1, \ldots, a_n\}\) with \(a = a_i\) for some \(i \in \{1, \ldots, n\}\).

In the first case \(I\) must satisfy \(\alpha^{\mathcal{I}}(a) = a^{\mathcal{I}} = a_i^{\mathcal{I}} = \alpha^{\mathcal{I}}(a_i)\) for two distinct individuals \(a\) and \(a_i\), which cannot be the case since we assumed unique names. In the second case \(I\) must satisfy \(\alpha^{\mathcal{I}}(a) = a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus \{\ldots, a_i^{\mathcal{I}}, \ldots\}\). In both cases, \(I\) cannot be a model of \(KB\), which contradicts \(S\) to contain an individual clash.
Let $S$ neither contain an inconsistency clash nor an individual clash. We construct an interpretation $I = (\Delta^I, \mathcal{I})$ from $S$ with $\Delta^I = N^S_1 \cup \{x \mid x \in N^S_2\}$, together with an $I$-assignment $\alpha^I : N^S_2 \cup N^S_3 \rightarrow \Delta^I$ that maps each variable $x$ in $S$ to a distinct individual $\iota_x$ not occurring in $S$, i.e. $\alpha^I(x) = \iota_x$ for $x \in N^S_2$.  

We define the interpretation function $\mathcal{I}$ such that an atomic concept $A$ is interpreted as $A^I = \{\alpha^I(o) \mid o : A \in S\}$ and a role $r$ as $r^I = \{(o_1, o_2) \mid (o_1, o_2) : r \in S\}$.

We show that $(I, \alpha^I)$ satisfies every constraint in $S$.

For any concept constraint $o : C$ in $S$ we show that $\alpha^I(o) \in C^I$ by induction on the structure of $C$. As base cases, we consider the concepts of the form $\neg A$ and $\{a_1, \ldots , a_k\}$, $\neg \{a_1, \ldots , a_k\}$, as well as roles. If $o : A$ is in $S$ then $\alpha^I(o) \in A^I$ by definition. If $o : \neg A$ is in $S$ then $o : A$ is not in $S$, since $S$ does not contain an inconsistency clash, and thus $\alpha^I(o) \notin A^I$. If $o : \{a_1, \ldots , a_k\}$ is in $S$ then $\alpha^I(o) \in \{a_1^I, \ldots , a_k^I\}$, since $S$ contains no individual clash: the $\neg \exists$-rule is not applicable, $x : \{a_1, \ldots , a_k\}$ cannot be in $S$. If $o : \neg \{a_1, \ldots , a_k\}$ is in $S$ then $\alpha^I(o) \notin \{a_1^I, \ldots , a_k^I\}$ and thus $\alpha^I(o) \in \Delta^I \setminus \{a_1^I, \ldots , a_k^I\}$, since $S$ contains no individual clash, while if $x : \neg \{a_1, \ldots , a_k\}$ is in $S$ then $\alpha^I(x) = \iota_x \in \Delta^I \setminus \{a_1^I, \ldots , a_k^I\}$ due to the unique name assumption. If $(o_1, o_2) : r$ is in $S$ then $(\alpha^I(o_1), \alpha^I(o_2)) \in r^I$ by definition. For the induction step, we subsequently consider the various complex forms of concepts. Observe that none of the tableau rules can be applied to $S$, as it is a completion.

- If $S$ contains $o : C \cap D$ then it contains both $o : C$ and $o : D$, since the $\cap$-rule cannot be applied. From the induction hypothesis we know that $S$ satisfies these two constraints, and thus, both $\alpha^I(o) \in C^I$ and $\alpha^I(o) \in D^I$ hold. Hence, $\alpha^I(o) \in (C^I \cap D^I) = (C \cap D)^I$ and $(I, \alpha^I)$ satisfies $o : C \cap D$.

- If $S$ contains $o : C \cup D$ then it contains either $o : C$ or $o : D$, since the $\cup$-rule cannot be applied. Without loss of generality, we can assume that $S$ contains $o : C$. From the induction hypothesis we know that $(I, \alpha^I)$ satisfies $o : C$, i.e. $\alpha^I(o) \in C^I$, and therefore we have that $\alpha^I(o) \in (C^I \cup D^I) = (C \cup D)^I$ and that $(I, \alpha^I)$ satisfies $o : C \cup D$.

- If $S$ contains $o_1 : \exists r.C$ then, since the $\exists$-rule cannot be applied, we have one of the following two cases: a) $S$ contains both $(o_1, o_2) : r$ and $o_2 : C$ for some $o_2 \in N^S_1 \cup N^S_2$, b) $o_1$ is blocked by some $o_3 \in N^S_1 \cup N^S_2$. In case a) we know from the induction hypothesis that $S$ satisfies these two constraints, and thus, both $(\alpha^I(o_1), \alpha^I(o_2)) \in r^I$ and $\alpha^I(o_2) \in D^I$ hold. Hence, $\alpha^I(o_1) \in (\exists r.C)^I$ and $S$ satisfies $o_1 : \exists r.C$. In case b) we know from the subset blocking condition that $S$ contains the constraint $o_3 : \exists r.C$, since $o_1$ is blocked by $o_3$. As $o_3$ cannot be blocked itself (see [5, Lemma 3.5]) and $S$ is complete, there are constraints $(o_3, o_2) : r$, $o_2 : C$ in $S$ for some $o_2 \in N^S_1 \cup N^S_2$. From the induction hypothesis we know that $(I, \alpha^I)$ satisfies these constraints, and thus, both $\alpha^I(o_3), \alpha^I(o_2) \in C^I$ hold. From the definition of $\mathcal{I}$ and the fact that $o_1$ is blocked by $o_3$ it follows that $(\alpha^I(o_1), \alpha^I(o_2)) \in C^I$, and hence, $(I, \alpha^I)$ satisfies $o_1 : \exists r.C$.

- If $S$ contains $o_1 : \forall r.C$ then it also contains $o_2 : C$ for every $o_2 \in N^S_1 \cup N^S_2$ with $(o_1, o_2) : r \in S$ , since the $\forall$-rule cannot be applied. In case $o_1$ is blocked by some $o_3 \in N^S_1 \cup N^S_2$, $S$ contains constraints $o_2 : C$ for every $o_2$ with $(o_3, o_2) : r \in S$, since it contains $o_3 : \forall r.C$ due to the subset blocking condition. By the definition of $\mathcal{I}$ and the fact that $o_3$ cannot be blocked itself (see [5, Lemma 3.5]), role instances $\{r(o_2) \mid o_2 \in S\}$ for all role instances $\{r(o_1), r(o_2)\}$ can only be induced by the role constraints $(o_1, o_2) : r$ and $(o_3, o_2) : r$, and from the induction hypothesis we know that these are satisfied by $S$ together with the respective concept constraints $o_2 : C$. Hence, we have that $\alpha^I(o_2) \in C^I$ for all role instances $\{r(o_1), r(o_2)\}$ and $S$ satisfies $o_1 : \forall r.C$.

By this, we have shown that $(I, \alpha^I)$ satisfies every constraint in $S$ and is therefore a solution for it.

Based on this correspondence between clash-free completions and their solutions, we can establish the correlation between solvability of constraint systems and the absence of preference clashes in their completions, as the main result of this report by the following proposition.

**Proposition 5 (circumscriptive ALCQ correctness).** Let $KB$ be an ALCQ knowledge base, $CP$ be a circumscription pattern and $S$ be the completion of a constraint system that contains at least the constraints of $S_{KB}$, with regard to $CP$ and $KB$. $S$ is $CP$-solvable with respect to $KB$ if and only if it contains no inconsistency clash, no individual clash and no preference clash with respect to $CP$ and $KB$.

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6 This construction is similar to the notion of a canonical interpretation as used in the literature such as [5].
Proof.
\(\Rightarrow\): Assume that \(S\) is \(\text{cp}\)-solvable with respect to \(KB\). According to Definition 4 there is a solution \((I, \alpha^J)\) for \(S\), such that \(I\) is a model of \(KB\) and there is no model \(J\) of \(KB\) with \(J <_{\text{cp}} I\). From Proposition 4, we know that \(S\) does neither contain an inconsistency clash nor an individual clash. We show by contradiction that \(S\) does also not contain a preference clash.

Assume that \(S\) contains a preference clash with respect to \(\text{cp}\) and \(KB\). Then, \(S_{KB}/\{/x\}\) has a completion \(S'\) with respect to \(\text{cp}\) and \(KB'\) that contains no inconsistency and no individual clash, where the knowledge base \(KB'\) is constructed based on \(\text{cp}\) and \(KB\) according to Algorithm 1. Observe that, by construction, \(KB \subseteq KB'\) and that \(\nu\) is a new individual in \(KB'\) that cannot occur in \(KB\). Hence, we have that \(S_{KB} \subseteq S_{KB'}/\{/x\} \subseteq S'\). Proposition 4(\(\Leftarrow\)) implies that there is a solution \((J, \alpha^J)\) for \(S'\), since \(S'\) is clash-free. Due to Lemma 2, and since \(S_{KB} \subseteq S'\), it follows that \(J\) is a model of both \(KB'\) and \(KB\). It remains to show that \(J <_{\text{cp}} I\), to contradict the containment of a preference clash in \(S\). Without loss of generality, we can assume that \(\Delta^J = \Delta^J\) and that \(a^J = a^J\) for all individuals \(a \in N_J\). We prove the following claims: a) \(A^J \subseteq \tilde{A}^J\) for all \(A \in M_{KB}\), and b) \(A^J \subseteq \tilde{A}^J\) for some \(A \in M_{KB}\).

- a) Due to the inclusion axioms for minimised concepts inserted into \(KB'\) by Algorithm 1, and since \(J\) is a model of \(KB'\), \(J\) has the property \(A^J \subseteq \{\alpha^J(a) \mid a \in S\} \) for each \(A \in M_{KB}\). For every \(A \in M_{KB}\), all the constraints \(a : A \in S\) are satisfied by \((I, \alpha^I)\), i.e., \(\alpha^I(a) \in \tilde{A}^J\), and therefore we have that \(\{\alpha^I(a) \mid a \in S\} \subseteq \tilde{A}^J\). Since \(\alpha^J\) and \(\alpha^I\) coincide on individuals, it follows that \(A^J \subseteq \tilde{A}^J\) for all \(A \in M_{KB}\).

- b) By construction of \(KB'\), \(S_{KB'}/\{/x\}\) contains a constraint \(x : \bigcup A \tilde{D}_A\) for one of the disjuncts \(D_A\), its completion \(S'\) contains a constraint of the form \(x : \{a_1, \ldots, a_n\} \cap \neg A\) with \(a_i : A \in S\) for \(i = 1 \ldots n\). Since \(S'\) is a completion to which none of the tableau rules apply, the \(\neg \rightarrow\) and the \(\neg \rightarrow\) have produced the constraints \(a : \{a_1, \ldots, a_n\}\) and \(a : \neg A\) in \(S'\) in which the variable \(x\) has been replaced by an individual \(a\). As a solution for \(S'\), \((J, \alpha^J)\) satisfies these two constraints and we have that both \(\alpha^J(a) \in (A^J \setminus \tilde{A}^J)\) and \(\alpha^J(a) \in \{\alpha^J(a) \mid a \in S\}\) hold. This implies that \(\alpha^J(a) \notin \tilde{A}^J\) and, since \((I, \alpha^I)\) satisfies the constraint \(a : \neg A\), that \(\alpha^J(x) = \alpha^I(a) \in \tilde{A}^J\). From the arguments under b) we already know that \(\tilde{A}^J \subseteq \tilde{A}^J\), and since we have an element \(\tilde{a}^J\) which is in \(\tilde{A}^J\) but not in \(\tilde{A}^J\), it follows that \(\tilde{A}^J \subset \tilde{A}^J\).

\(\Leftarrow\): Let \(S\) contain no clash. From Proposition 4 we know that there is a solution \((I, \alpha^J)\) for \(S\). We show by contradiction that there is no model \(J\) of \(KB\) such that \(J <_{\text{cp}} I\).

Assume that there is a model \(J\) of \(KB\) with \(J <_{\text{cp}} I\). First we show that for some \(J\)-assignment \(\alpha^J\), \((J, \alpha^J)\) is a solution for \(S_{KB'}/\{/x\}\), where the knowledge base \(KB'\) is constructed according to Algorithm 1. Due to \(J <_{\text{cp}} I\) we know that \(\Delta^J = \Delta^J\) and \(a^J = a^J\) for all individuals \(a \in \Delta^J\), and that for some \(A \in M_{KB}\) there is an element \(\nu \in \Delta^J\) which is in \(\tilde{A}^J\) but not in \(\tilde{A}^J\). Let \(\alpha^J\) be a \(J\)-assignment with \(\alpha^J(x) = \nu\). Since \(J\) is a model of \(KB\), \((J, \alpha^J)\) is a solution for \(S_{KB}\) due to Lemma 2. Moreover, as the individual \(\nu\) is new to \(KB'\) and \(KB \subseteq KB'\) by construction of \(KB'\), the replacement of \(\nu\) by \(x\) does not affect any constraint in \(S_{KB}\) and we have that \(S_{KB} \subseteq S_{KB'}/\{/x\}\). Hence, it suffices to show that the constraints in \(S_{KB'}/\{/x\}\) \(\setminus S_{KB}\) are satisfied by \((J, \alpha^J)\). For this purpose, we consider the axioms in \(KB' \setminus KB\) that are inserted into \(KB'\) by Algorithm 1, and that can be a) concept inclusion axioms of the form \(A \subseteq \{a_1, \ldots, a_n\}\), or b) the concept assertion axiom \((\bigcup A \tilde{D}_A)(i)\) with disjuncts \(D_A\) of the form \(\neg A \sqcap \{a_1, \ldots, a_n\}\), for individuals \(\{a_i : A \in S\}\) with \(i \in \{1, \ldots, n\}\).

- a) For every \(A \in M_{KB}\), \(KB'\) contains an axiom \(\tilde{A} \subseteq \{a_i, \ldots, a_n\}\) with individuals \(a_i\) that occur in concept constraints of the form \(A \subseteq \{a_i\}\). Since \(S\) is a completion, in any constraint of the form \(x : \tilde{A}\) the variable \(x\) has been replaced by an individual \(a \in N_Y\) in \(S\) due to the \(\neg \rightarrow\) rule, such that for any constraint \(x : \tilde{A}\) we have that \(x = a_i\) for some \(i \in \{1, \ldots, n\}\). Since \(I\) is a solution for \(S\), we have that \(\tilde{A}^J \subseteq \{\alpha^J(a_1), \ldots, \alpha^J(a_n)\} = \{a^J_1, \ldots, a^J_n\}\). Since \(\tilde{A}^J \subseteq \tilde{A}^J\) holds by assumption, \(J\) satisfies \(\tilde{A}^J \subseteq \{a^J_1, \ldots, a^J_n\}\), and thus, the axiom \(\tilde{A} \subseteq \{a_1, \ldots, a_n\}\) for every \(\tilde{A} \in M_{KB}\). If there are no assertions \(\tilde{A}(a_i)\) in \(KB'\) then \(\tilde{A}^J = \emptyset\) and the respective axiom has the form \(A \subseteq \bot\). Hence, \((J, \alpha^J)\) satisfies all the constraints \(\forall x : C\) that result from these inclusion axioms in \(S_{KB'}/\{/x\}\).

- b) Furthermore, due to the concept assertion \((\bigcup A \tilde{D}_A)(i)\) in \(KB'\), \(S_{KB'}/\{/x\}\) contains the constraint \(x : \bigcup A \tilde{D}_A\) with disjuncts \(D_A\) of the form \(\neg A \sqcap \{a_1, \ldots, a_n\}\). Since from b) we know that \(\tilde{A}^J \subseteq \tilde{A}^J\)
\{a_1, \ldots, a_n\}, and since \(a^F = a^D\) for all individuals \(a\), we get that \(\tilde{A}^F = \{a_1^F, \ldots, a_n^F\} \subseteq \Delta^D\). As for some \(A \in M_{KB}\) the element \(\iota^F\) is in \(\tilde{A}^F\) but not in \(\tilde{A}^J\), we have that \(\alpha^F_{x,a}(x) \in \tilde{A}^F \setminus \tilde{A}^J\), and thus, \(\alpha_{x,a}^J(x) = (\{a_1^F, \ldots, a_n^F\} \setminus \tilde{A}^J) \cap \{\alpha_{x,a}^F(a_1), \ldots, \alpha_{x,a}^F(a_k)\}\). Hence, the pair \((J, \alpha^J_{x,a})\) satisfies the constraint \(x : \bigcup_{\tilde{A}} D_{\tilde{A}}\) for some \(A \in M_{KB}\) with \(\tilde{A}^J \subseteq \tilde{A}^F\), as one of its disjuncts is satisfied.

Having shown that \((J, \alpha^J_{x,a})\) is a solution for \(S_{KB}[t/x]\), from Proposition 3(\(\Rightarrow\)) and from Proposition 4(\(\Rightarrow\)) it follows that there is a clash-free completion of \(S_{KB}[^{\dagger}\neg x]\). Hence, \(S\) must contain a preference clash, which contradicts the existence of \(J\).

3.4 Sound and Complete Reasoning in Circumscriptive ALCO

Finally, we can show that the presented calculus provides an effective procedure for reasoning with circumscribed knowledge bases, by the following theorems.

**Theorem 1 (soundness).** Let \(KB\) be an ALCO knowledge base, \(cp\) a circumscription pattern and \(C\) an ALCO concept. If the repeated application of the tableaux rules results in a clash-free completion of the constraint system \(S_{KB} \cup \{x : C\}\) then \(C\) is satisfiable with respect to \(circ_{CP}(KB)\).

**Proof.** Let \(S\) be the clash-free completion of \(S_{KB} \cup \{x : C\}\) produced by the tableaux procedure. Due to Proposition 5(\(\Leftarrow\)), \(S\) is \(cp\)-solvable with respect to \(KB\), and due to Proposition 3(\(\Leftarrow\)), \(S_{KB} \cup \{x : C\}\) also is. Finally, due to Proposition 2(\(\Leftarrow\)), \(C\) is satisfiable with respect to \(circ_{CP}(KB)\).

**Theorem 2 (completeness).** Let \(KB\) be an ALCO knowledge base, \(cp\) a circumscription pattern and \(C\) an ALCO concept. If \(C\) is satisfiable with respect to \(circ_{CP}(KB)\) then some repeated application of the tableaux rules results in a clash-free completion of the constraint system \(S_{KB} \cup \{x : C\}\).

**Proof.** Let \(C\) be satisfiable with respect to \(circ_{CP}(KB)\). Due to Proposition 2(\(\Rightarrow\)), the constraint system \(S_{KB} \cup \{x : C\}\) is \(cp\)-solvable with respect to \(KB\), and due to Lemma 3 the tableaux procedure results in a completion \(S'\) of \(S_{KB} \cup \{x : C\}\). From Proposition 3(\(\Rightarrow\)), we know that \(S'\) is also \(cp\)-solvable with respect to \(KB\). Finally, due to Proposition 5(\(\Rightarrow\)), \(S'\) does not contain any clash.

By Theorem 1 and Theorem 2, the proposed tableaux calculus is a sound and complete decision procedure for ALCO with concept circumscription.

4 Conclusion

We have presented a tableaux calculus for concept satisfiability with respect to circumscribed DL knowledge bases in the logic ALCO. Building on tableaux procedures for classical DLs, the calculus checks a constraint system not only for clashes due to inconsistent concept assertion and individual naming, but also for preference clashes, which occur whenever the model associated with the constraint system is not minimal with respect to the preference relation \(<_{cp}\). This check is performed by testing a specifically constructed classical ALCO knowledge base for satisfiability, which requires reasoning in classical DL with nominals and equality between individuals.

We have proved that the presented calculus is a decision procedure for concept satisfiability in circumscriptive ALCO, to which other reasoning tasks can be reduced. By this we have devised a first practical algorithmisation for description logic with circumscription that integrates well with state of the art tableaux methods for DL reasoning. This lays a basis for further investigations on optimisation of the calculus within the framework of tableaux procedures as a guided way for model construction. We have implemented a first prototype \(^7\) of the calculus in Java that works together with ontology development tools, such as Protege\(^8\), via the DIG\(^9\) interface. Notice also that our algorithm does not behave worse than the theoretical worst-case complexity, which was shown to be NExp\(^{NP}\)-complete in \([4]\). In our case, the call to the oracle corresponds to the calls to an ALCO-reasoner for checking

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\(^7\) Available at http://www.fzi.de/downloads/wim/sgr/CircDL.zip.

\(^8\) http://protege.stanford.edu/

\(^9\) http://dl.kr.org/dig/interface.html
the preference clash, which is Exp-complete and thus as good as possible, since Exp is not known to be different from NP.

As future work we see the update of the calculus to support more expressive features, such as prioritisation between minimised concepts or the remaining constructs of the Web Ontology Language OWL [13] and its latest version OWL 1.1. Moreover, optimisation issues need to be addressed to obtain a more efficient reasoning procedure. First ideas for specific optimisations would be to employ model caching techniques for the inner classical tableaux step as $\mathcal{KB}'$ might be identical in multiple cases, to postpone assertions of individuals to minimised predicates to avoid constructing non-minimal models, and to exploit early closing of tableaux branches through preference clash detection. Besides these, it would be interesting to see how well preferential tableaux performs when included in optimised state-of-the-art DL reasoners. Furthermore, as an open line for research it remains to develop a methodology for the formulation of appropriate circumscription patterns in various cases of non-monotonic reasoning, as pointed out in [4].

References


[^10]: http://owl1_1.cs.manchester.ac.uk/