Integrating Logic Programs and Connectionist Systems

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"Logic is everywhere ..."
Introduction & Motivation: Overview

- Introduction & Motivation
- Propositional Logic
  - Existing Approaches
  - Propositional Logic Programs and the Core Method
- First-Order Logic
  - Existing Approaches
  - First-Order Logic Programs and the Core Method
- The Neural-Symbolic Learning Cycle
- Challenge Problems
Introduction & Motivation: Connectionist Systems

- Well-suited to learn, to adapt to new environments, to degrade gracefully etc.
- Many successful applications.
- Approximate functions.
  - Hardly any knowledge about the functions is needed.
  - Trained using incomplete data.
- Declarative semantics is not available.
- Recursive networks are hardly understood.
- McCarthy 1988: We still observe a propositional fixation.
- Structured objects are difficult to represent.
  - Smolensky 1987: Can we instantiate the power of symbolic computation within fully connectionist systems?
Introduction & Motivation: Logic Systems

- Well-suited to represent and reason about structured objects and structure-sensitive processes.
- Many successful applications.
- Direct implementation of relations and functions.
- Explicit expert knowledge is required.
- Highly recursive structures.
- Well understood declarative semantics.
- Logic systems are brittle.
- Expert knowledge may not be available.

Can we instantiate the power of connectionist computation within a logic system?
Introduction & Motivation: Objective

▶ Seek the best of both paradigms!
▶ Understanding the relation between connectionist and logic systems.
▶ Contribute to the open research problems of both areas.
▶ Well developed for propositional case.
▶ Hard problem: going beyond.
▶ In this lecture:
  ▶ Overview on existing approaches.
  ▶ Logic programs and recurrent networks.
  ▶ Semantic operators for logic programs can be computed by connectionist systems.
Connectionist Networks

- A connectionist network consists of
  - a set $U$ of units and
  - a set $W \subseteq U \times U$ of connections.

- Each connection is labeled by a weight $w \in \mathbb{R}$.
- If there is a connection from unit $u_j$ to $u_k$, then $w_{kj}$ is its associated weight.
- A unit is specified by
  - an input vector $\vec{i} = (i_1, \ldots, i_m)$, $i_j \in \mathbb{R}$, $1 \leq j \leq m$,
  - an activation function $\Phi$ mapping $\vec{i}$ to a potential $p \in \mathbb{R}$,
  - an output function $\Psi$ mapping $p$ to an (output) value $v \in \mathbb{R}$.

- If there is a connection from $u_j$ to $u_k$ then $w_{kj}v_j$ is the input received by $u_k$ along this connection.
- The potential and value of a unit are synchronously recomputed (or updated).
- Often a linear time $t$ is added as parameter to input, potential and value.
- The state of a network with units $u_1, \ldots, u_n$ at time $t$ is $(v_1(t), \ldots, v_n(t))$. 
Propositional Logic

▶ Existing Approaches

▷ Finite Automata and McCulloch-Pitts Networks
▷ Weighted Automata and Semiring Artificial Neural Networks
▷ Propositional Reasoning and Symmetric/Stochastic Networks
▷ Other Approaches

▶ Propositional Logic Programs and the Core Method

▷ The Very Idea
▷ Logic Programs
▷ Propositional Core Method
▷ Backpropagation
▷ Knowledge-Based Artificial Neural Networks
▷ Propositional Core Method using Sigmoidal Units
▷ Further Extensions
Finite Automata and McCulloch-Pitts Networks

- **McCulloch, Pitts 1943:**
  Can the activities of nervous systems be modelled by a logical calculus?
- A **McCulloch-Pitts network** consists of a set $U$ of binary threshold units and a set $W \subseteq U \times U$ of weighted connections.
- The set $U_I$ of input units is defined as $U_I = \{ u_k \in U \mid (\forall u_j \in U) w_{kj} = 0 \}$.
- The set $U_O$ of output units is defined as $U_O = \{ u_j \in U \mid (\forall u_k \in U) w_{kj} = 0 \}$.

**Theorem** McCulloch-Pitts networks are finite automata and vice versa.
Binary Threshold Units

- $u_k$ is a binary threshold unit if

$$\Phi(\vec{i}_k) = p_k = \sum_{j=1}^{m} w_{kj} v_j$$
$$\Psi(p_k) = v_k = \begin{cases} 1 & \text{if } p_k \geq \theta_k \\ 0 & \text{otherwise} \end{cases}$$

where $\theta_k \in \mathbb{R}$ is a threshold.

- Three binary threshold units:

1. $v_1, w_{21} = -1, \theta_2 = -0.5, v_2 = \neg v_1$
2. $v_1, w_{31} = 1, \theta_3 = 0.5, v_3 = v_1 \lor v_2$
3. $v_1, w_{31} = 1, \theta_3 = 1.5, v_3 = v_1 \land v_2$
Bader, Hölldobler, Scalzitti 2004:
Can the result by McCulloch and Pitts be extended to weighted automata?

Let \((K, \oplus, \odot, 0_K, 1_K)\) be a semiring.

\(u_k\) is a \(\oplus\)-unit if
\[
\Phi(\vec{v}_k) = p_k = \bigoplus_{j=1}^{m} w_{kj} \odot v_j
\]
\[
\Psi(p_k) = v_k = p_k
\]

\(u_k\) is a \(\odot\)-unit if
\[
\Phi(\vec{v}_k) = p_k = \bigodot_{j=1}^{m} w_{kj} \odot v_j
\]
\[
\Psi(p_k) = v_k = p_k
\]

A semiring artificial neural network consists of a set \(U\) of \(\oplus\)- and \(\odot\)-units and a set \(W \subseteq U \times U\) of \(K\)-weighted connections.

Theorem  Weighted automata are semiring artificial neural networks.
Symmetric Networks

▶ **Hopfield 1982**: Can statistical models for magnetic materials explain the behavior of certain classes of networks?

▶ **A symmetric network** consists of a set $U$ of binary threshold units and a set $W \subseteq U \times U$ of weighted connections such that $w_{kj} = w_{jk}$ for all $k, j$ with $k \neq j$.

▶ Asynchronous update procedure:
while state $\vec{v}$ is unstable: update an arbitrary unit.

▶ Minimizes the energy function $E(\vec{v}) = - \sum_{k<j} w_{kj}v_j v_k + \sum_k \theta_k v_k$.
Stochastic Networks or Boltzmann Machines

- **Hinton, Sejnowski 1983**: Can we escape local minima?
- A **stochastic network** is a symmetric network, but the values are computed probabilistically

\[
P(v_k = 1) = \frac{1}{1 + e^{(\theta_k - p_k)/T}}
\]

where \( T \) is called **pseudo temperature**.

- In equilibrium stochastic networks are more likely to be in a state with low energy.
- **Kirkpatrick et al. 1983**: Can we compute a global minima?
- **Simulated annealing**: decrease \( T \) gradually.
- **Theorem (Geman, Geman 1984)**
  A global minima is reached if \( T \) is decreased in infinitesimal small steps.
Propositional Reasoning and Energy Minimization

- **Pinkas 1991:**
  Is there a link between propositional logic and symmetric networks?

- Let $D = \langle C_1, \ldots, C_m \rangle$ be a propositional formula in clause form.

- We define

  $$
  \tau(C) = \begin{cases}
  0 & \text{if } C = [], \\
  p & \text{if } C = [p], \\
  1 - p & \text{if } C = [\neg p], \\
  \tau(C_1) + \tau(C_2) - \tau(C_1)\tau(C_2) & \text{if } C = (C_1 \lor C_2).
  \end{cases}
  $$

  $$
  \tau(D) = \sum_{i=1}^{m} (1 - \tau(C_i))
  $$

- **Example**

  $$
  \tau(\langle [\neg o, m], [\neg s, \neg m], [\neg c, m], [\neg c, s], [\neg v, \neg m] \rangle) = vm - cm - cs + sm - om + 2c + o.
  $$
Propositional Reasoning and Symmetric Networks

- **Theorem** \( \vec{v} \models D \) iff \( \tau(D) \) has a global minima at \( \vec{v} \).
- **Compare**
  \[
  \tau(D) = \nu m - \nu m - \nu s + \nu \nu m - \nu m + 2\nu + \nu
  \]
  with
  \[
  E(\vec{v}) = -\sum_{k<j} \nu w_{kji}\nu_j \nu_k + \sum_k \theta_k \nu_k.
  \]
Propositional Non-Monotonic Reasoning

- Pinkas 1991a:
  Can the above mentioned approach be extended to non-monotonic reasoning?

- Consider $D = \langle (C_1, k_1), \ldots, (C_m, k_m) \rangle$, where $C_i$ are clauses and $k_i \in \mathbb{R}^+$.

- The penalty of $\vec{v}$ for $(C, k)$ is $k$ if $\vec{v} \not\models C$ and 0 otherwise.

- The penalty of $\vec{v}$ for $D$ is the sum of the penalties for $(C_i, k_i)$.

- $\vec{v}$ is preferred over $\vec{w}$ wrt $D$ if the penalty of $\vec{v}$ for $D$ is smaller than the penalty of $\vec{w}$ for $D$.

- Modify $\tau$ to become $\tau(D) = \sum_{i=1}^{m} k_i (1 - \tau(C_i))$, e.g.,

  \[
  \tau(\langle ([\neg o, m], 1), ([\neg s, \neg m], 2), ([\neg c, m], 4), ([\neg c, s], 4), ([\neg v, \neg m], 4) \rangle)
  = 4vm - 4cm - 4cs + 2sm - om + 8c + o.
  \]

- The corresponding stochastic network computes most preferred interpretations.
Propositional Logic Programs and the Core Method

- The Very Idea
- Logic Programs
- Propositional Core Method
- Backpropagation
- Knowledge-Based Artificial Neural Networks
- Propositional Core Method using Sigmoidal Units
- Further Extensions
The Very Idea

- Various semantics for logic programs coincide with fixed points of associated immediate consequence operators (e.g., Apt, vanEmden 1982).

- **Banach Contraction Mapping Theorem** A contraction mapping $f$ defined on a complete metric space $(X, d)$ has a unique fixed point. The sequence $y, f(y), f(f(y)), \ldots$ converges to this fixed point for any $y \in X$.

  - **Fitting 1994**: Consider logic programs, whose immediate consequence operator is a contraction.

- **Funahashi 1989**: Every continuous function on the reals can be uniformly approximated by feedforward connectionist networks.

  - **Hölldobler, Kalinke, Störr 1999**: Consider logic programs, whose immediate consequence operator is continuous on the reals.
Metrics

A metric on a space \( M \) is a mapping \( d : M \times M \to \mathbb{R} \) such that

- \( d(x, y) = 0 \) iff \( x = y \),
- \( d(x, y) = d(y, x) \), and
- \( d(x, y) \leq d(x, z) + d(z, y) \).

Let \((M, d)\) be a metric space and \( S = (s_i \mid s_i \in M) \) a sequence.

- \( S \) converges if \((\exists s \in M)(\forall \epsilon > 0)(\exists N)(\forall n \geq N) d(s_n, s) \leq \epsilon \).
- \( S \) is Cauchy if \((\forall \epsilon > 0)(\exists N)(\forall n, m \geq N) d(s_n, s_m) \leq \epsilon \).
- \((M, d)\) is complete if every Cauchy sequence converges.

A mapping \( f : M \to M \) is a contraction on \((M, d)\) if \((\exists 0 < k < 1)(\forall x, y \in M) d(f(x), f(y)) \leq k \cdot d(x, y) \).
Propositional Logic Programs

- A propositional logic program \( P \) over a propositional language \( \mathcal{L} \) is a finite set of clauses

\[
A \leftarrow L_1 \land \ldots \land L_n,
\]

where \( A \) is an atom, \( L_i \) are literals and \( n \geq 0 \).

\( P \) is definite if all \( L_i, 1 \leq i \leq n \) are atoms.

- Let \( \mathcal{V} \) be the set of all propositional variables occurring in \( \mathcal{L} \).

- An interpretation \( I \) is a mapping \( \mathcal{V} \rightarrow \{\top, \bot\} \).

- \( I \) can be represented by the set of atoms which are mapped to \( \top \) under \( I \).

- \( 2^\mathcal{V} \) is the set of all interpretations.

- Immediate consequence operator \( T_P : 2^\mathcal{V} \rightarrow 2^\mathcal{V} : \)

\[
T_P(I) = \{ A \mid \text{there is a clause } A \leftarrow L_1 \land \ldots \land L_n \in P \text{ such that } I \models L_1 \land \ldots \land L_n \}.
\]

- \( I \) is a supported model iff \( T_P(I) = I \).
The Core Method

- Let $\mathcal{L}$ be a logic language.
- Given a logic program $\mathcal{P}$ together with immediate consequence operator $T_{\mathcal{P}}$.
- Let $\mathcal{I}$ be the set of interpretations for $\mathcal{P}$.
- Find a mapping $R : \mathcal{I} \rightarrow \mathbb{R}^n$.
- Construct a feed-forward network computing $f_{\mathcal{P}} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, called the core, such that the following holds:
  - If $T_{\mathcal{P}}(I) = J$ then $f_{\mathcal{P}}(R(I)) = R(J)$, where $I, J \in \mathcal{I}$.
  - If $f_{\mathcal{P}}(\vec{s}) = \vec{t}$ then $T_{\mathcal{P}}(R^{-1}(\vec{s})) = R^{-1}(\vec{t})$, where $\vec{s}, \vec{t} \in \mathbb{R}^n$.
- Connect the units in the output layer recursively to the units in the input layer.
- Show that the following holds
  - $I = \text{lfp}(T_{\mathcal{P}})$ iff the recurrent network converges to or approximates $R(I)$.

Connectionist model generation using recurrent networks with feed forward core.
3-Layer Recurrent Networks

At each point in time all units do:

- apply activation function to obtain potential,
- apply output function to obtain output.
Propositional Core Method using Binary Threshold Units

- Let \( \mathcal{L} \) be the language of propositional logic over a set \( \mathcal{V} \) of variables.
- Let \( \mathcal{P} \) be a propositional logic program, e.g.,

\[
\mathcal{P} = \{ A, C \leftarrow A \land \neg B, C \leftarrow \neg A \land B \}.
\]

- \( \mathcal{I} = 2^\mathcal{V} \) is the set of interpretations for \( \mathcal{P} \).
- \( T_\mathcal{P}(I) = \{ A \mid A \leftarrow L_1 \land \ldots \land L_m \in \mathcal{P} \text{ such that } I \models L_1 \land \ldots \land L_m \} \).

\[
\begin{align*}
T_\mathcal{P}(\emptyset) &= \{ A \} \\
T_\mathcal{P}(\{ A \}) &= \{ A, C \} \\
T_\mathcal{P}(\{ A, C \}) &= \{ A, C \} = \text{lfp}(T_\mathcal{P})
\end{align*}
\]
Representing Interpretations

- $\mathcal{I} = 2^\mathcal{V}$
- Let $n = |\mathcal{V}|$ and identify $\mathcal{V}$ with $\{1, \ldots, n\}$.
- Define $R : \mathcal{I} \rightarrow \mathbb{R}^n$ such that for all $1 \leq j \leq n$ we find:

$$R(I)[j] = \begin{cases} 1 & \text{if } j \in I, \\ 0 & \text{if } j \not\in I. \end{cases}$$

E.g., if $\mathcal{V} = \{A, B, C\} = \{1, 2, 3\}$ and $I = \{A, C\}$ then $R(I) = (1, 0, 1)$.
- Other encodings are possible, e.g.,

$$R(I)[j] = \begin{cases} 1 & \text{if } j \in I, \\ -1 & \text{if } j \not\in I. \end{cases}$$
Computing the Core

- Consider again $\mathcal{P} = \{ A, C \leftarrow A \land \neg B, C \leftarrow \neg A \land B \}$.
- A translation algorithm translates $\mathcal{P}$ into a core of binary threshold units:

```
  A   B   C
  \frac{3}{2} \frac{3}{2} \frac{3}{2}
  \frac{3}{2} \frac{3}{2} \frac{3}{2}
  \frac{1}{2} \frac{1}{2} \frac{1}{2}
```

- Input layer
- Hidden layer
- Output layer
Some Results

- **Proposition** 2-layer networks cannot compute $T_P$ for definite $P$.
- **Theorem** For each program $P$, there exists a core computing $T_P$.
- **Recall** $P = \{A, \ C \leftarrow A \land \lnot B, \ C \leftarrow \lnot A \land B\}$.
- **Adding recurrent connections:**
More Results

▶ A logic program $\mathcal{P}$ is said to be strongly determined if there exists a metric $d$ on the set of all Herbrand interpretations for $\mathcal{P}$ such that $T_{\mathcal{P}}$ is a contraction wrt $d$.

▶ Corollary Let $\mathcal{P}$ be a strongly determined program. Then there exists a core with recurrent connections such that the computation with an arbitrary initial input converges and yields the unique fixed point of $T_{\mathcal{P}}$.

▶ Let $n$ be the number of clauses and $m$ be the number of propositional variables occurring in $\mathcal{P}$.

- $2m + n$ units, $2mn$ connections in the core.
- $T_{\mathcal{P}}(I)$ is computed in 2 steps.
- The parallel computational model to compute $T_{\mathcal{P}}(I)$ is optimal.
- The recurrent network settles down in $3n$ steps in the worst case.

Rule Extraction (1)

▶ Proposition

For each core $C$ there exists a program $\mathcal{P}$ such that $C$ computes $T_\mathcal{P}$.

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Rule Extraction (2)

Extracted program:

\[
P = \{ A_1 \leftarrow \neg A_1 \land \neg A_2, \\
      A_1 \leftarrow \neg A_1 \land A_2, \\
      A_1 \leftarrow A_1 \land \neg A_2, \\
      A_1 \leftarrow A_1 \land A_2, \\
      A_2 \leftarrow \neg A_1 \land \neg A_2, \\
      A_2 \leftarrow \neg A_1 \land A_2, \\
      A_2 \leftarrow A_1 \land \neg A_2, \\
      A_2 \leftarrow A_1 \land A_2 \}
\]

Simplified form:

\[
P = \{ A_1, A_2 \leftarrow A_1, A_2 \leftarrow \neg A_1 \land A_2 \}
\]
3-Layer Feed-Forward Networks Revisited

▶ Theorem (Funahashi 1989) Suppose that $\Psi : \mathbb{R} \to \mathbb{R}$ is non-constant, bounded, monotone increasing and continuous. Let $K \subseteq \mathbb{R}^n$ be compact, let $f : K \to \mathbb{R}$ be continuous, and let $\varepsilon > 0$. Then there exists a 3-layer feed-forward network with output function $\Psi$ for the hidden layer and linear output function for the input and output layer whose input-output mapping $\bar{f} : K \to \mathbb{R}$ satisfies

$$\max_{x \in K} |f(x) - \bar{f}(x)| < \varepsilon.$$ 

▶ Every continuous function $f : K \to \mathbb{R}$ can be uniformly approximated by input-output functions of 3-layer feed-forward networks.

▶ $u_k$ is a sigmoidal unit if

$$\Phi(\vec{i}_k) = p_k = \sum_{j=1}^{m} w_{kj} v_j$$

$$\Psi(p_k) = v_k = \frac{1}{1 + e^{\beta(\theta_k - p_k)}}$$

where $\theta_k \in \mathbb{R}$ is a threshold (or bias) and $\beta > 0$ a steepness parameter.
Backpropagation

- Training set of input-output pairs \( \{(\vec{i}^l, \vec{o}^l) \mid 1 \leq l \leq n\} \).
- Minimize \( E = \sum_l E^l \) where \( E^l = \frac{1}{2} \sum_k (o_k^l - v_k^l)^2 \).
- Gradient descent algorithm to learn appropriate weights.
- Backpropagation

1. Initialize weights arbitrarily.
2. Present input pattern \( \vec{i}^l \) at time \( t \).
3. Compute output pattern \( \vec{v}^l \) at time \( t + 2 \).
4. Change weights according to \( \Delta w_{ij}^l = \eta \delta_i^l v_j^l \), where

   - \( \delta_i^l = \begin{cases} 
   \Psi'_i(p_i^l) \times (o_i^l - v_i^l) & \text{if } i \text{ is output unit,} \\
   \Psi'_i(p_i^l) \times \sum_k \delta_k^l w_{ki} & \text{if } i \text{ is hidden unit,}
   \end{cases} \)

   - \( \eta > 0 \) is called learning rate.
Knowledge Based Artificial Neural Networks

- **Towell, Shavlik 1994**: Can we do better than empirical learning?
- Sets of hierarchical logic programs, e.g.,

\[ \mathcal{P} = \{ A \leftarrow B \land C \land \neg D, A \leftarrow D \land \neg E, H \leftarrow F \land G, K \leftarrow A, \neg H \}. \]
Knowledge Based Artificial Neural Networks – Learning

- Given hierarchical sets of propositional rules as background knowledge.
- Map rules into multi-layer feed forward networks with sigmoidal units.
- Add hidden units (optional).
- Add units for known input features that are not referenced in the rules.
- Fully connect layers.
- Add near-zero random numbers to all links and thresholds.
- Apply backpropagation.

- Empirical evaluation: system performs better than purely empirical and purely hand-built classifiers.
Knowledge Based Artificial Neural Networks – A Problem

- Works if rules have few conditions and there are few rules with the same head.

\[ p_A = p_B = 9\omega \quad \text{and} \quad v_A = v_B = \frac{1}{1 + e^{\beta(9.5\omega - 9\omega)}} \approx 0.46 \quad \text{with} \quad \beta = 1. \]

\[ p_C = 0.92\omega \quad \text{and} \quad v_c = \frac{1}{1 + e^{\beta(0.5\omega - 0.92\omega)}} \approx 0.6 \quad \text{with} \quad \beta = 1. \]
Propositional Core Method using Pipolar Sigmoidal Units

- d’Avila Garcez, Zaverucha, Carvalho 1997:
  Can we combine the ideas in Hölldobler, Kalinke 1994 and Towell, Shavlik 1994 while avoiding the above mentioned problem?
- Consider propositional logic language.
- Let $I$ be an interpretation and $a \in [0, 1]$.

$$R(I)[j] = \begin{cases} v \in [a, 1] & \text{if } j \in I, \\ w \in [-1, -a] & \text{if } j \notin I. \end{cases}$$

- Replace threshold and sigmoidal units by bipolar sigmoidal ones, i.e., units with

$$\Phi(\vec{i}_k) = p_k = \sum_{j=1}^m w_{kj}v_j, \quad \Psi(p_k) = v_k = \frac{2}{1 + e^{\beta(\theta_k - p_k)}} - 1,$$

where $\theta_k \in \mathbb{R}$ is a threshold (or bias) and $\beta > 0$ a steepness parameter.
The Task

- How should $a$, $\omega$ and $\theta_i$ be selected such that:
  - $v_i \in [a, 1]$ or $v_i \in [-1, -a]$ and
  - the core computes the immediate consequence operator?
Hidden Layer Units

Consider \( A \leftarrow L_1 \land \ldots \land L_n \).

Let \( u \) be the hidden layer unit for this rule.

- Suppose \( I \models L_1 \land \ldots \land L_n \).
  - \( u \) receives input \( \geq \omega a \) from unit representing \( L_i \).
  - \( p_u \geq n \omega a = p_u^+ \).

- Suppose \( I \not\models L_1 \land \ldots \land L_n \).
  - \( u \) receives input \( \leq -\omega a \) from at least one unit representing \( L_i \).
  - \( p_u \leq (n - 1) \omega 1 - \omega a = p_u^- \).

\[ \theta_u = \frac{n \omega a + (n - 1) \omega - \omega a}{2} = (na + n - 1 - a)\frac{\omega}{2} = (n - 1)(a + 1)\frac{\omega}{2}. \]
Output Layer Units

- Let $\mu$ be the number of clause with head $A$.
- Consider $A \leftarrow L_1 \land \ldots \land L_n$.
- Suppose $I \models L_1 \land \ldots \land L_n$.
  \[ p_A \geq \omega a + (\mu - 1)\omega(-1) = \omega a - (\mu - 1)\omega = p_A^+ \]
- Suppose for all rules of the form $A \leftarrow L_1 \land \ldots \land L_n$ we find $I \not\models L_1 \land \ldots \land L_n$.
  \[ p_A \leq -\mu\omega a = p_A^- \]
- $\theta_A = \frac{\omega a - (\mu - 1)\omega - \mu a}{2} = (a - \mu + 1 - \mu a)\frac{\omega}{2} = (1 - \mu)(a + 1)\frac{\omega}{2}$. 

Propositional Logic Programs and the Core Method
Computing a Value for $a$

$\triangleright p_u^+ > p_u^-:
\triangleright n\omega a > (n - 1)\omega - \omega a.$
\triangleright n\omega a + \omega a > (n - 1)\omega.$
\triangleright a(n + 1)\omega > (n - 1)\omega.$
\triangleright a > \frac{n-1}{n+1}.$

$\triangleright p_A^+ > p_A^-:
\triangleright \omega a - (\mu - 1)\omega > -\mu a\omega.$
\triangleright \omega a + \mu a\omega > (\mu - 1)\omega.$
\triangleright a(1 + \mu)\omega > (\mu - 1)\omega.$
\triangleright a > \frac{\mu-1}{\mu+1}.$

$\triangleright$ Consider all rules $\rightsquigarrow$ minimum value for $a$. 

Propositional Logic Programs and the Core Method
Computing a Value for $\omega$

$\Psi(p) = \frac{2}{1 + e^{\beta(\theta - p)}} - 1 \geq a.$

$\frac{2}{1 + e^{\beta(\theta - p)}} \geq 1 + a.$

$\frac{2}{1 + a} \geq 1 + e^{\beta(\theta - p)}.$

$\frac{2}{1 + a} - 1 = \frac{2}{1 + a} - \frac{1 + a}{1 + a} = \frac{1 - a}{1 + a} \geq e^{\beta(\theta - p)}.$

$\ln\left(\frac{1 - a}{1 + a}\right) \geq \beta(\theta - p).$

$\frac{1}{\beta} \ln\left(\frac{1 - a}{1 + a}\right) \geq \theta - p.$

Consider a hidden layer unit:

$\frac{1}{\beta} \ln\left(\frac{1 - a}{1 + a}\right) \geq (n - 1)(a + 1)\frac{\omega}{2} - n\omega a = \frac{na + n - a - 1 - 2na}{2} \omega = \frac{n - 1 - a(n + 1)}{2} \omega.$

$\omega \geq \frac{2}{(n - 1 - a(n + 1))\beta} \ln\left(\frac{1 - a}{1 + a}\right)$ because $a \geq \frac{n - 1}{n + 1}.$

Consider all hidden and output layer units as well as the case that $\Psi(p) \leq -a$: minimum value for $\omega.$
Results

- Relation to logic programs is preserved.
- The core is trainable by backpropagation.
- Many interesting applications, e.g.:
  - DNA sequence analysis.
  - Power system fault diagnosis.
- Empirical evaluation:
  system performs better than well-known machine learning systems.
- See d’Avila Garcez, Broda, Gabbay 2002 for details.
Further Extensions

- Many-valued logic programs
- Modal logic programs
- Answer set programming
- Metalevel priorities
- Rule extraction
Propositional Core Method – Three-Valued Logic Programs

- **Kalinke 1994:** Consider truth values $\top$, $\bot$, $u$.
- Interpretations are pairs $I = \langle I^+, I^- \rangle$.
- Immediate consequence operator $\Phi_P(I) = \langle J^+, J^- \rangle$, where

$$
\begin{align*}
J^+ &= \{ A \mid A \leftarrow L_1 \land \ldots \land L_m \in \mathcal{P} \text{ and } I(L_1 \land \ldots \land L_m) = \top \}, \\
J^- &= \{ A \mid \text{for all } A \leftarrow L_1 \land \ldots \land L_m \in \mathcal{P} : I(L_1 \land \ldots \land L_m) = \bot \}.
\end{align*}
$$

- Let $n = |\mathcal{V}|$ and identify $\mathcal{V}$ with $\{1, \ldots, n\}$.
- Define $R : \mathcal{I} \to \mathbb{R}^{2n}$ as follows:

\[
R(I)[2j - 1] = \begin{cases} 
1 & \text{if } j \in I^+ \\
0 & \text{if } j \notin I^+ 
\end{cases} \quad \text{and} \quad R(I)[2j] = \begin{cases} 
1 & \text{if } j \in I^- \\
0 & \text{if } j \notin I^- 
\end{cases}
\]
Propositional Core Method – Multi-Valued Logic Programs

For each program $\mathcal{P}$, there exists a core computing $\Phi_{\mathcal{P}}$, e.g.,

$$\mathcal{P} = \{ C \leftarrow A \land \neg B, \ D \leftarrow C \land E, \ D \leftarrow \neg C \}. $$

Lane, Seda 2004: Extension to finitely determined sets of truth values.
Propositional Core Method – Modal Logic Programs

- Let $\mathcal{L}$ be a propositional logic language plus
  - the modalities $\Box$ and $\Diamond$, and
  - a finite set of labels $w_1, \ldots, w_k$ denoting worlds.
- Let $B$ be an atom, then $\Box B$ and $\Diamond B$ are modal atoms.
- A modal definite logic program $\mathcal{P}$ is a set of clauses of the form
  \[ w_i : A \leftarrow A_1 \land \ldots \land A_m \]
  together with a finite set of relations $w_i \triangleright w_j$, where
  $w_i, 1 \leq i, j \leq k$, are labels and $A, A_1, \ldots, A_m$ are atoms or modal atoms.
- $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$, where $\mathcal{P}_i$ consists of all clauses labelled with $w_i$. 
Modal Logic Programs – Semantics

Example: $\mathcal{P} = \{w_1 : A, w_1 : \Diamond C \leftarrow A\} \cup \{w_2 : B\} \cup \{w_3 : B\} \cup \{w_4 : B\} \cup \{w_1 \triangleright w_2, w_1 \triangleright w_3, w_1 \triangleright w_4, w_2 \triangleright w_4, \}$

Kripke semantics:

$\Diamond C \quad \Box C \\
\Diamond C \quad \Box C \\
\Diamond B \quad \Box B \\
\Box B \quad \Box A \\
\Box B \quad \Box A$

$\mathcal{G} = (W, R, F)$

$\mathcal{F}(C \{w_1\}) = w_4$

$\mathcal{F}(C \{w_1\}) = w_4$
Modal Immediate Consequence Operator

- Interpretations are tuples $I = \langle I_1, \ldots, I_k \rangle$
- Immediate consequence operator $MT_P(I) = \langle J_1, \ldots, J_k \rangle$, where

$$J_i = \begin{array}{l}
\{ A \mid \text{there exists } A \leftarrow A_1 \land \ldots \land A_m \in \mathcal{P}_i \\
\text{such that } \{A_1, \ldots, A_m\} \subseteq I_i \} \\
\cup \{ \Diamond A \mid \text{there exists } w_i \triangleright w_j \in \mathcal{P} \text{ and } A \in I_j \} \\
\cup \{ \Box A \mid \text{for all } w_i \triangleright w_j \in \mathcal{P} \text{ we find } A \in I_j \} \\
\cup \{ A \mid \text{there exists } w_j \triangleright w_i \in \mathcal{P} \text{ and } \Box A \in I_j \} \\
\cup \{ A \mid \text{there exists } w_j \triangleright w_i \in \mathcal{P}, \Diamond A \in I_j \text{ and } f_A(w_j) = w_i \} \end{array}$$
Modal Logic Programs – The Translation Algorithm

- Let \( n = |\mathcal{V}| \) and identify \( \mathcal{V} \) with \( \{1, \ldots, n\} \).
- Let \( a \in [0, 1] \).
- Define \( R : \mathcal{I} \rightarrow \mathbb{R}^{3n} \) as follows:

\[
R(I)[3j - 2] = \begin{cases} 
  v \in [a, 1] & \text{if } j \in I_j \\
  w \in [-1, -a] & \text{if } j \notin I_j
\end{cases}
\]

\[
R(I)[3j - 1] = \begin{cases} 
  v \in [a, 1] & \text{if } \Box j \in I_j \\
  w \in [-1, -a] & \text{if } \Box j \notin I_j
\end{cases}
\]

\[
R(I)[3j] = \begin{cases} 
  v \in [a, 1] & \text{if } \Diamond j \in I_j \\
  w \in [-1, -a] & \text{if } \Diamond j \notin I_j
\end{cases}
\]

- Translation algorithm such that

  - for each world the “local” part of \( MT_{\mathcal{P}} \) is computed by a core,
  - the cores are turned into recurrent networks, and
  - the cores are connected with respect to the given set of relations.
The Example Network

Propositional Logic Programs and the Core Method
First-Order Logic

► Existing Approaches

► Reflexive Reasoning and SHRUTI
► Connectionist Term Representations
  ● Holographic Reduced Representations Plate 1991
  ● Recursive Auto-Associative Memory Pollack 1988
► Horn logic and CHCL Hölldobler 1990, Hölldobler, Kurfess 1992
► Other Approaches

► First-Order Logic Programs and the Core Method

► Initial Approach
► Construction of Approximating Networks
► Topological Analysis and Generalisations
► Employing Iterated Function Systems
Reflexive Reasoning

- Humans are capable of performing a wide variety of cognitive tasks with extreme ease and efficiency.
- For traditional AI systems, the same problems turn out to be intractable.
- Human consensus knowledge: about $10^8$ rules and facts.
- Wanted: “Reflexive” decisions within sublinear time.
- Shastri, Ajjanagadde 1993: SHRUTI.
SHRUTI – Knowledge Base

▶ Finite set of constants $C$, finite set of variables $\mathcal{V}$.

▶ Rules:

$$(\forall X_1 \ldots X_m)(p_1(\ldots) \land \ldots \land p_n(\ldots)) \rightarrow (\exists Y_1 \ldots Y_k p(\ldots)).$$

$\exists \ p, \ p_i, 1 \leq i \leq n$, are multi-place predicate symbols.

$\exists \ Arguments of the \ p_i: \ variables \ from \ \{X_1, \ldots, X_m\} \subseteq \mathcal{V}.$

$\exists \ Arguments of \ p \ are \ from \ \{X_1, \ldots, X_m\} \cup \{Y_1, \ldots, Y_k\} \cup C.$

$\exists \ \{Y_1, \ldots, Y_k\} \subseteq \mathcal{V}.$

$\exists \ \{X_1, \ldots, X_m\} \cap \{Y_1, \ldots, Y_k\} = \emptyset.$

▶ Facts and queries (goals):

$$\exists Z_1 \ldots Z_l \ q(\ldots).$$

$\exists \ Multi-place \ predicate \ symbol \ q.$

$\exists \ Arguments of \ q \ are \ from \ \{Z_1, \ldots, Z_l\} \cup C.$

$\exists \ \{Z_1, \ldots, Z_l\} \subseteq \mathcal{V}.$
Further Restrictions

- Restrictions to rules, facts, and goals:
  - No function symbols except constants.
  - Only universally bound variables may occur as arguments in the conditions of a rule.
  - All variables occurring in a fact or goal occur only once and are existentially bound.
  - An existentially quantified variable is only unified with variables.
  - A variable which occurs more than once in the conditions of a rule must occur in the conclusion of the rule and must be bound when the conclusion is unified with a goal.
  - A rule is used only a fixed number of times.

Incompleteness.
SHRUTI – Example

Rules
\[ \mathcal{P} = \{ \begin{align*}
    & \text{owns}(Y, Z) \leftarrow \text{gives}(X, Y, Z), \\
    & \text{owns}(X, Y) \leftarrow \text{buys}(X, Y), \\
    & \text{can-sell}(X, Y) \leftarrow \text{owns}(X, Y), \\
    & \text{gives}(\text{john}, \text{josephine}, \text{book}), \\
    & (\exists X) \text{buys}(\text{john}, X), \\
    & \text{owns}(\text{josephine}, \text{ball})
\} \]

Queries:
\[ \begin{align*}
    & \text{can-sell}(\text{josephine}, \text{book}) \rightarrow \text{yes} \\
    & (\exists X) \text{owns}(\text{josephine}, X) \rightarrow \begin{cases} 
        \text{yes} & \{X \mapsto \text{book}\} \\
        & \{X \mapsto \text{ball}\}
    \end{cases}
\]
SHRUTI : The Network
Solving the Variable Binding Problem

First-Order Logic
SHRUTI – Remarks

- Answers are derived in time proportional to depth of search space.
- Number of units as well as of connections is linear in the size of the knowledge base.
- Extensions:
  - compute answer substitutions
  - allow a fixed number of copies of rules
  - allow multiple literals in the body of a rule
  - built in a taxonomy

- Biological plausibility.
- Trading expressiveness for time and size.
- Logical reconstruction by Beringer, Hölldobler 1993:
  - Reflexive reasoning is reasoning by reduction.
First-Order Logic Programs and the Core Method

- Initial Approach
- Construction of Approximating Networks
- Topological Analysis and Generalisations
- Employing Iterated Function Systems
Logic Programs

- A logic program $\mathcal{P}$ over a first-order language $\mathcal{L}$ is a finite set of clauses

$$A \leftarrow L_1 \land \ldots \land L_n,$$

where $A$ is an atom, $L_i$ are literals and $n \geq 0$.

- $B_{\mathcal{L}}$ is the set of all ground atoms over $\mathcal{L}$ called Herbrand base.

- A Herbrand interpretation $I$ is a mapping $B_{\mathcal{L}} \rightarrow \{\top, \bot\}$.

- $2^{B_{\mathcal{L}}}$ is the set of all Herbrand interpretations.

- ground($\mathcal{P}$) is the set of all ground instances of clauses in $\mathcal{P}$.

- Immediate consequence operator $T_\mathcal{P} : 2^{B_{\mathcal{L}}} \rightarrow 2^{B_{\mathcal{L}}}$:

$$T_\mathcal{P}(I) = \{ A \mid \text{there is a clause } A \leftarrow L_1 \land \ldots \land L_n \in \text{ground}(\mathcal{P}) \text{ such that } I \models L_1 \land \ldots \land L_n \}.$$

- $I$ is a supported model iff $T_\mathcal{P}(I) = I$. 

First-Order Logic Programs and the Core Method
The Initial Approach

- Hölldobler, Kalinke, Störr 1999:
  Can the core method be extended to first-order logic programs?

- Problem

  - Given a logic program $\mathcal{P}$ over a first order language $\mathcal{L}$ together with $T_{\mathcal{P}} : 2^{B_{\mathcal{L}}} \rightarrow 2^{B_{\mathcal{L}}}$.
  - $B_{\mathcal{L}}$ is countably infinite.
  - The method used to relate propositional logic and connectionist systems is not applicable.
  - How can the gap between the discrete, symbolic setting of logic, and the continuous, real valued setting of connectionist networks be closed?
The Goal

- Find $R : 2^B \rightarrow \mathbb{R}$ and $f_P : \mathbb{R} \rightarrow \mathbb{R}$ such that the following conditions hold.

  - $T_P(I) = I'$ implies $f_P(R(I)) = R(I')$.
  - $f_P(x) = x'$ implies $T_P(R^{-1}(x)) = R^{-1}(x')$.

  $\Rightarrow f_P$ is a sound and complete encoding of $T_P$.

  - $T_P$ is a contraction on $2^B$ iff $f_P$ is a contraction on $\mathbb{R}$.

  $\Rightarrow$ The contraction property and fixed points are preserved.

  - $f_P$ is continuous on $\mathbb{R}$.

  $\Rightarrow$ A connectionist network approximating $f_P$ is known to exist.
Acyclic Logic Programs

Let \( \mathcal{P} \) be a program over a first order language \( \mathcal{L} \).

A level mapping for \( \mathcal{P} \) is a function \( l : B_\mathcal{L} \rightarrow \mathbb{N} \).

- We define \( l(\neg A) = l(A) \).

We can associate a metric \( d_\mathcal{L} \) with \( \mathcal{L} \) and \( l \). Let \( I, J \in 2^{B_\mathcal{L}} \):

\[
d_\mathcal{L}(I, J) = \begin{cases} 
0 & \text{if } I = J \\
2^{-n} & \text{if } n \text{ is the smallest level on which } I \text{ and } J \text{ differ.}
\end{cases}
\]

- **Proposition (Fitting 1994)** \( (2^{B_\mathcal{L}}, d_\mathcal{L}) \) is a complete metric space.

- \( \mathcal{P} \) is said to be acyclic wrt a level mapping \( l \), if for every \( A \leftarrow L_1 \land \ldots \land L_n \in \text{ground}(\mathcal{P}) \) we find \( l(A) > l(L_i) \) for all \( i \).

- **Proposition** Let \( \mathcal{P} \) be an acyclic logic program wrt \( l \) and \( d_\mathcal{L} \) the metric associated with \( \mathcal{L} \) and \( l \), then \( T_\mathcal{P} \) is a contraction on \( (2^{B_\mathcal{L}}, d_\mathcal{L}) \).
Mapping Interpretations to Real Numbers

- Let $D = \{ r \in \mathbb{R} \mid r = \sum_{i=1}^{\infty} a_i 4^{-i}, \text{ where } a_i \in \{0, 1\} \text{ for all } i \}$.  
- Let $l$ be a bijective level mapping.  
- $\{\top, \bot\}$ can be identified with $\{0, 1\}$.  
- The set of all mappings $B_L \rightarrow \{\top, \bot\}$ can be identified with the set of all mappings $\mathbb{N} \rightarrow \{0, 1\}$.  
- Let $I_L$ be the set of all mappings from $B_L$ to $\{0, 1\}$.  
- Let $R : I_L \rightarrow D$ be defined as  
  $$R(I) = \sum_{i=1}^{\infty} I(l^{-1}(i)) 4^{-i}.$$  
- Proposition $R$ is a bijection.

We have a sound and complete encoding of interpretations.
We define $f_P : \mathcal{D} \rightarrow \mathcal{D} : r \mapsto R(T_P(R^{-1}(r)))$.

We have a sound and complete encoding of $T_P$.

**Proposition** Let $P$ be an acyclic program wrt a bijective level mapping. $f_P$ is a contraction on $\mathcal{D}$.

Contraction property and fixed points are preserved.
Approximating Continuous Functions

- **Corollary** $f_P$ is continuous.

- **Recall Funahashi’s theorem:**
  
  > Every continuous function $f : K \rightarrow \mathbb{R}$ can be uniformly approximated by input-output functions of 3-layer feed forward networks.

- **Theorem** $f_P$ can be uniformly approximated by input-output functions of 3-layer feed forward networks.

  > $T_P$ can be approximated as well by applying $R^{-1}$.

Connectionist network approximating immediate consequence operator exists.
An Example

Consider $P = \{q(0), q(s(X)) \leftarrow q(X)\}$ and let $l(q(s^n(0))) = n + 1$.

- $P$ is acyclic wrt $l$, $l$ is bijective, $R(B_L) = \frac{1}{3}$.
- $f_P(R(I)) = 4^{-l(q(0))} + \sum_{q(X) \in I} 4^{-l(q(s(X)))}
  = 4^{-l(q(0))} + \sum_{q(X) \in I} 4^{-l(q(X)) + 1} = \frac{1 + R(I)}{4}$.

Approximation of $f_P$ to accuracy $\varepsilon$ yields

$$f^\varepsilon(x) \in \left[\frac{1 + x}{4} - \varepsilon, \frac{1 + x}{4} + \varepsilon\right].$$

Starting with some $x$ and iterating $f^\varepsilon$ yields in the limit a value

$$r \in \left[\frac{1 - 4\varepsilon}{3}, \frac{1 + 4\varepsilon}{3}\right].$$

Applying $R^{-1}$ to $r$ we find

$$q(s^n(0)) \in R^{-1}(r) \text{ if } n < -\log_4 \varepsilon - 1.$$
Approximation of Interpretations

- Let $\mathcal{P}$ be a logic program over a first order language $\mathcal{L}$ and $l$ a level mapping.
- An interpretation $I$ approximates an interpretation $J$ to a degree $n \in \mathbb{N}$ if for all atoms $A \in B_\mathcal{L}$ with $l(A) < n$ we find $I(A) = \top$ iff $J(A) = \top$.

\[
I \text{ approximates } J \text{ to a degree } n \text{ iff } d_\mathcal{L}(I, J) \leq 2^{-n}.
\]
Approximation of Supported Models

- Given an acyclic logic program $\mathcal{P}$ with bijective level mapping.
- Let $T_P$ be the immediate consequence operator associated with $\mathcal{P}$ and $M_P$ the least supported model of $\mathcal{P}$.
- We can approximate $T_P$ by a 3-layer feed forward network.
- We can turn this network into a recurrent one.

Does the recurrent network approximate the supported model of $\mathcal{P}$?

**Theorem** For an arbitrary $m \in \mathbb{N}$ there exists a recursive network with sigmoidal activation function for the hidden layer units and linear activation functions for the input and output layer units computing a function $\overline{f}_P$ such that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in [-1, 1]$ we find

$$d_L(R^{-1}(\overline{f}_P^n(x)), M_P) \leq 2^{-m}.$$
Literature


