Integrating Logic Programs and Connectionist Systems

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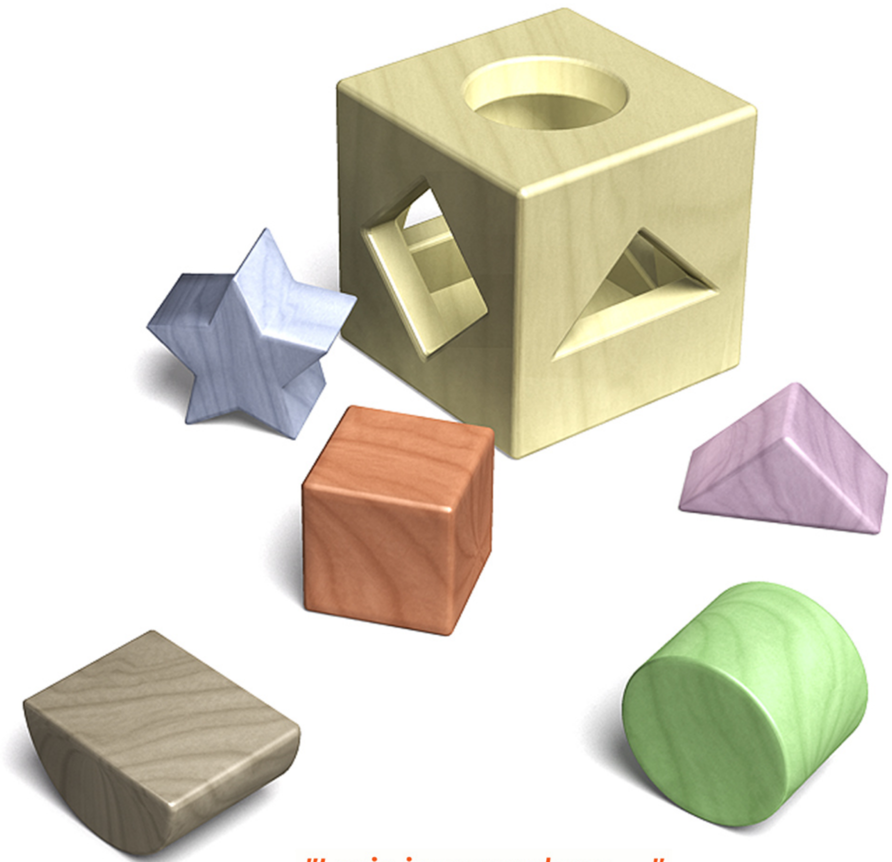


"Logic is everywhere ..."

Introduction & Motivation: Overview

- Introduction & Motivation
- Propositional Logic
  - Existing Approaches
  - Propositititonal Logic Programs and the Core Method
- First-Order Logic
  - Existing Approaches
  - First-Order Logic Programs and the Core Method
- The Neural-Symbolic Learning Cycle
- Challenge Problems

Introduction & Motivation: Connectionist Systems

- Well-suited to learn, to adapt to new environments, to degrade gracefully etc.
- Many successful applications.
- Approximate functions.
  - Hardly any knowledge about the functions is needed.
  - Trained using incomplete data.
- Declarative semantics is not available.
- Recursive networks are hardly understood.
- McCarthy 1988: We still observe a propositional fixation.
- Structured objects are difficult to represent.
  - Smolensky 1987: Can we instantiate the power of symbolic computation within fully connectionist systems?

Introduction & Motivation: Logic Systems

- Well-suited to represent and reason about structured objects and structure-sensitive processes.
- Many successful applications.
- Direct implementation of relations and functions.
- Explicit expert knowledge is required.
- Highly recursive structures.
- Well understood declarative semantics.
- Logic systems are brittle.
- Expert knowledge may not be available.
    - Can we instantiate the power of connectionist computation within a logic system?

Introduction & Motivation: Objective

- Seek the best of both paradigms!
- Understanding the relation between connectionist and logic systems.
- Contribute to the open research problems of both areas.
- Well developed for propositional case.
- Hard problem: going beyond.
- In this lecture:
  - Overview on existing approaches.
  - Logic programs and recurrent networks.
  - Semantic operators for logic programs can be computed by connectionist systems.

Introduction to the core method.

Connectionist Networks

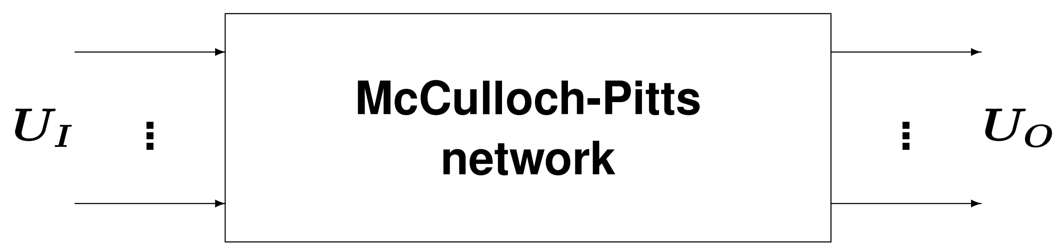
- A connectionist network consists of
  - a set U of units and
  - a set $W \subseteq U \times U$ of connections.
- Each connection is labeled by a weight $w \in \mathbb{R}$.
- If there is a connection from unit u_j to u_k, then w_{kj} is its associated weight.
- A unit is specified by
  - an input vector $\vec{i} = (i_1, \ldots, i_m)$, $i_j \in \mathbb{R}$, $1 \leq j \leq m$,
  - an activation function Φ mapping $\vec{i}$ to a potential $p \in \mathbb{R}$,
  - an output function Ψ mapping p to an (output) value $v \in \mathbb{R}$.
- If there is a connection from u_j to u_k
  then $w_{kj}v_j$ is the input received by u_k along this connection.
- The potential and value of a unit are synchronously recomputed (or updated).
- Often a linear time t is added as parameter to input, potential and value.
- The state of a network with units $u_1, \ldots, u_n$ at time t is $(v_1(t), \ldots, v_n(t))$.

Propositional Logic

- Existing Approaches
  - Finite Automata and McCulloch-Pitts Networks
  - Weighted Automata and Semiring Artificial Neural Networks
  - Propositional Reasoning and Symmetric/Stochastic Networks
  - Other Approaches
- Propositititonal Logic Programs and the Core Method
  - The Very Idea
  - Logic Programs
  - Propositional Core Method
  - Backpropagation
  - Knowledge-Based Artificial Neural Networks
  - Propositional Core Method using Sigmoidal Units
  - Further Extensions

Finite Automata and McCulloch-Pitts Networks

- **McCulloch, Pitts 1943:**
  Can the activities of nervous systems be modelled by a logical calculus?
- **A McCulloch-Pitts network consists of a set U of binary threshold units and a set $W \subseteq U \times U$ of weighted connections.**
- **The set U_I of input units is defined as $U_I = \{u_k \in U \mid (\forall u_j \in U)\, w_{kj} = 0\}$.**
- **The set U_O of output units is defined as $U_O = \{u_j \in U \mid (\forall u_k \in U)\, w_{kj} = 0\}$.**



- **Theorem McCulloch-Pitts networks are finite automata and vice versa.**

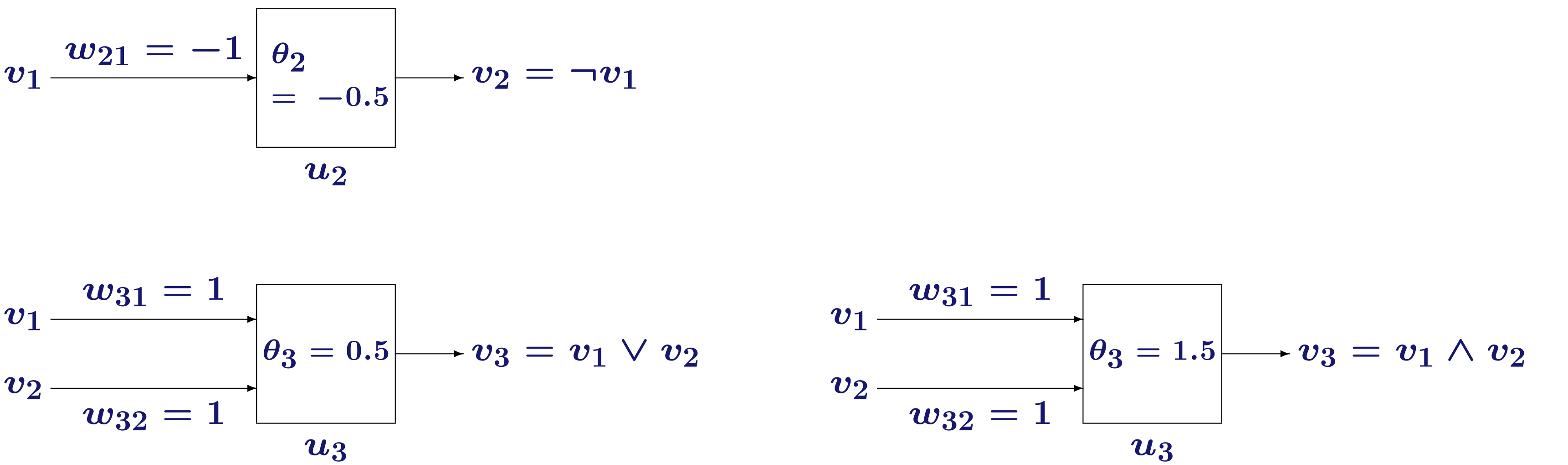
Binary Threshold Units

- u_k is a **binary threshold unit** if

$$\begin{array}{rcccl} \Phi(\vec{i}_k) & = & p_k & = & \sum_{j=1}^{m} w_{kj} v_j \\ \Psi(p_k) & = & v_k & = & \begin{cases} 1 & \text{if } p_k \geq \theta_k \\ 0 & \text{otherwise} \end{cases} \end{array}$$

where $\theta_k \in \mathbb{R}$ is a **threshold**.

- Three binary threshold units:



Weighted Automata and Semiring Artificial Neural Networks

- **Bader, Hölldobler, Scalzitti 2004:**
  Can the result by McCulloch and Pitts be extended to weighted automata?
- Let $(K, \oplus, \odot, 0_K, 1_K)$ be a semiring.
- u_k is a $\oplus$-unit if

$$\begin{array}{lllll} \Phi(\vec{i}_k) & = & p_k & = & \bigoplus_{j=1}^m w_{kj} \odot v_j \\ \Psi(p_k) & = & v_k & = & p_k \end{array}$$

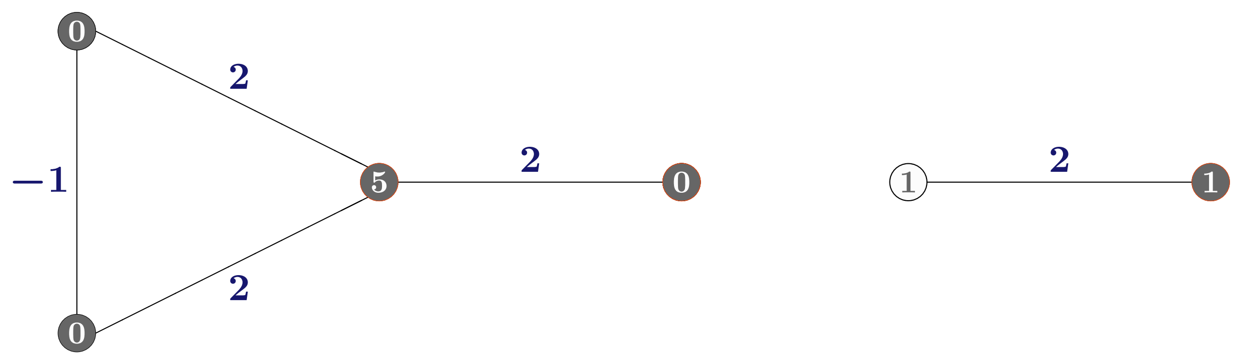
- u_k is a $\odot$-unit if

$$\begin{array}{lllll} \Phi(\vec{i}_k) & = & p_k & = & \bigodot_{j=1}^m w_{kj} \odot v_j \\ \Psi(p_k) & = & v_k & = & p_k \end{array}$$

- A semiring artificial neural network consists of a set U of $\oplus$- and $\odot$-units and a set $W \subseteq U \times U$ of K-weighted connections.
- Theorem Weighted automata are semiring artificial neural networks.

Symmetric Networks

- **Hopfield 1982: Can statistical models for magnetic materials explain the behavior of certain classes of networks?**
- **A symmetric network consists of a set U of binary threshold units and a set $W \subseteq U \times U$ of weighted connections such that $w_{kj} = w_{jk}$ for all k, j with $k \neq j$.**
- **Asynchronous update procedure: while state $\vec{v}$ is unstable: update an arbitrary unit.**



- **Minimizes the energy function $E(\vec{v}) = -\sum_{k<j} w_{kj} v_j v_k + \sum_k \theta_k v_k$.**

Stochastic Networks or Boltzmann Machines

- **Hinton, Sejnowski 1983: Can we escape local minima?**
- **A stochastic network is a symmetric network, but the values are computed probabilistically**

$$P(v_k = 1) = \frac{1}{1 + e^{(\theta_k - p_k)/T}}$$

where T is called pseudo temperature.

- **In equilibrium stochastic networks are more likely to be in a state with low energy.**
- **Kirkpatrick etal. 1983: Can we compute a global minima?**
- **Simulated annealing: decrease T gradually.**
- **Theorem (Geman, Geman 1984)**
  A global minima is reached if T is decreased in infinitesimal small steps.

Propositional Reasoning and Energy Minimization

- **Pinkas 1991:**
  Is there a link between propositional logic and symmetric networks?
- Let $D = \langle C_1, \ldots, C_m \rangle$ be a propositional formula in clause form.
- We define

$$\tau(C) = \begin{cases} 0 & \text{if } C = [\,], \\ p & \text{if } C = [p], \\ 1 - p & \text{if } C = [\neg p], \\ \tau(C_1) + \tau(C_2) - \tau(C_1)\tau(C_2) & \text{if } C = (C_1 \vee C_2). \end{cases}$$

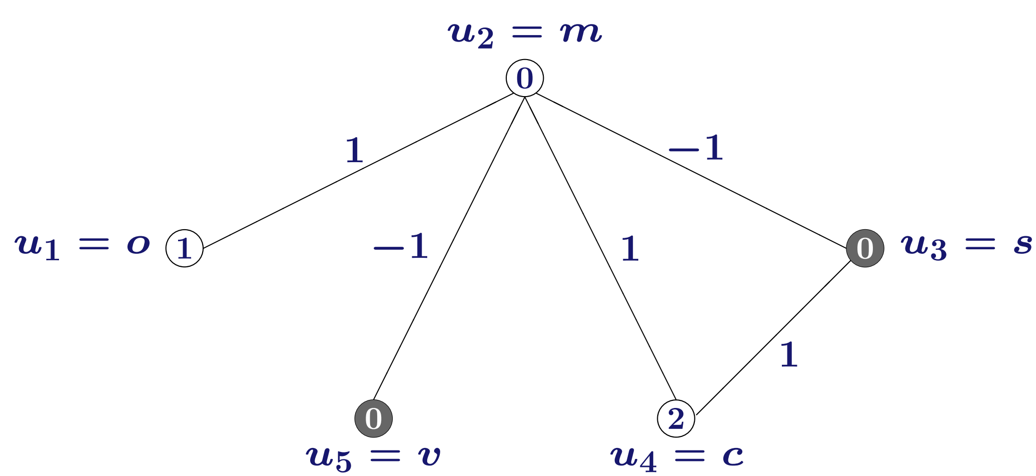
$$\tau(D) = \sum_{i=1}^{m} (1 - \tau(C_i))$$

- Example

$$\begin{aligned} & \tau(\langle [\neg o, m],\ [\neg s, \neg m],\ [\neg c, m],\ [\neg c, s],\ [\neg v, \neg m] \rangle) \\ = \ & vm - cm - cs + sm - om + 2c + o. \end{aligned}$$

Propositional Reasoning and Symmetric Networks

- **Theorem** $\vec{v} \models D$ **iff** $\tau(D)$ **has a global minima at** $\vec{v}$.
- **Compare** $\tau(D) = vm - cm - cs + sm - om + 2c + o$
  with $E(\vec{v}) = -\sum_{k<j} w_{kj} v_j v_k + \sum_k \theta_k v_k$.



Propositional Non-Monotonic Reasoning

- **Pinkas 1991a:**
  Can the above mentioned approach be extended to non-monotonic reasoning?
- Consider $D = \langle (C_1, k_1), \ldots, (C_m, k_m) \rangle$, where C_i are clauses and $k_i \in \mathbb{R}^+$.
- The penalty of $\vec{v}$ for (C, k) is k if $\vec{v} \not\models C$ and 0 otherwise.
- The penalty of $\vec{v}$ for D is the sum of the penalties for (C_i, k_i).
- $\vec{v}$ is preferred over $\vec{w}$ wrt D
  if the penalty of $\vec{v}$ for D is smaller than the penalty of $\vec{w}$ for D.
- Modify τ to become $\tau(D) = \sum_{i=1}^{m} k_i(1 - \tau(C_i))$, e.g.,

$$\tau(\langle ([\neg o, m], 1),\ ([\neg s, \neg m], 2),\ ([\neg c, m], 4),\ ([\neg c, s], 4),\ ([\neg v, \neg m], 4) \rangle) \\ = 4vm - 4cm - 4cs + 2sm - om + 8c + o.$$

- The corresponding stochastic network computes most preferred interpretations.

Propositiitonal Logic Programs and the Core Method

- The Very Idea
- Logic Programs
- Propositional Core Method
- Backpropagation
- Knowledge-Based Artificial Neural Networks
- Propositional Core Method using Sigmoidal Units
- Further Extensions

The Very Idea

- Various semantics for logic programs coincide with fixed points of associated immediate consequence operators (e.g., Apt, vanEmden 1982).
- Banach Contraction Mapping Theorem A contraction mapping f defined on a complete metric space (X, d) has a unique fixed point. The sequence $y,\ f(y),\ f(f(y)),\ \ldots$ converges to this fixed point for any $y \in X$.
  - Fitting 1994: Consider logic programs, whose immediate consequence operator is a contraction.
- Funahashi 1989: Every continuous function on the reals can be uniformly approximated by feedforward connectionist networks.
  - Hölldobler, Kalinke, Störr 1999: Consider logic programs, whose immediate consequence operator is continuous on the reals.

Metrics

- A metric on a space M is a mapping $d : M \times M \longrightarrow \mathbb{R}$ such that
  - $d(x, y) = 0$ iff $x = y$,
  - $d(x, y) = d(y, x)$, and
  - $d(x, y) \leq d(x, z) + d(z, y)$.
- Let (M, d) be a metric space and $S = (s_i \mid s_i \in M)$ a sequence.
  - S converges if $(\exists s \in M)(\forall \epsilon > 0)(\exists N)(\forall n \geq N)\, d(s_n, s) \leq \epsilon$.
  - S is Cauchy if $(\forall \epsilon > O)(\exists N)(\forall n, m \geq N)\, d(s_n, s_m) \leq \epsilon$.
  - (M, d) is complete if every Cauchy sequence converges.
- A mapping $f : M \longrightarrow M$ is a contraction on (M, d)
  if $(\exists 0 < k < 1)(\forall x, y \in M)\, d(f(x), f(y)) \leq k \cdot d(x, y)$.

Propositional Logic Programs

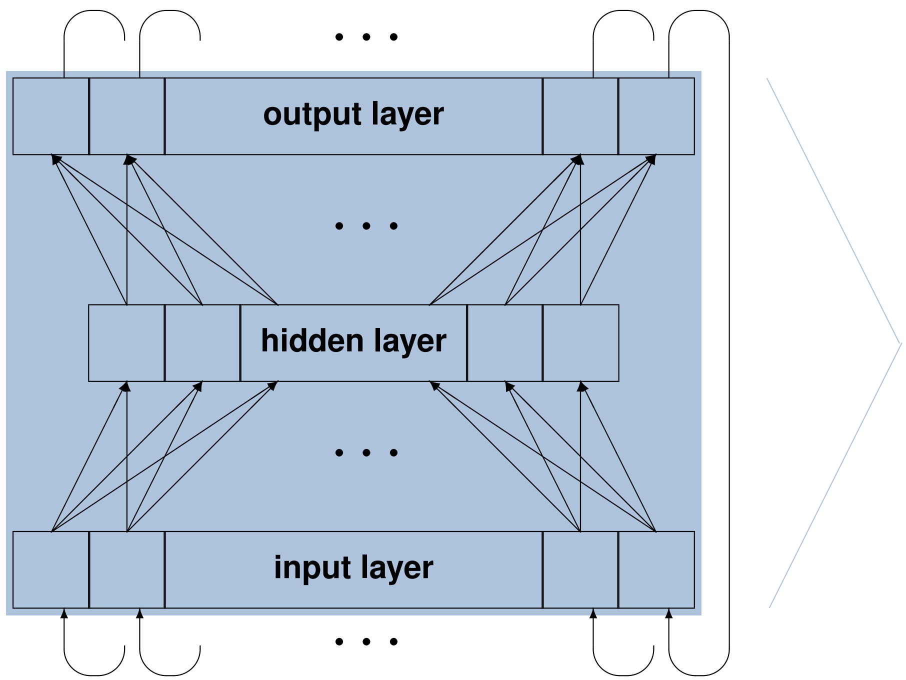
- A propositional logic program $\mathcal{P}$ over a propositional language $\mathcal{L}$ is a finite set of clauses
$$A \leftarrow L_1 \wedge \ldots \wedge L_n,$$
where A is an atom, L_i are literals and $n \geq 0$.
$\mathcal{P}$ is definite if all L_i, $1 \leq i \leq n$ are atoms.
- Let $\mathcal{V}$ be the set of all propositional variables occurring in $\mathcal{L}$.
- An interpretation I is a mapping $\mathcal{V} \rightarrow \{\top, \bot\}$.
- I can be represented by the set of atoms which are mapped to $\top$ under I.
- $2^{\mathcal{V}}$ is the set of all interpretations.
- Immediate consequence operator $T_{\mathcal{P}} : 2^{\mathcal{V}} \rightarrow 2^{\mathcal{V}}$:
$$T_{\mathcal{P}}(I) = \{A \mid \text{there is a clause } A \leftarrow L_1 \wedge \ldots \wedge L_n \in P \text{ such that } I \models L_1 \wedge \ldots \wedge L_n\}.$$
- I is a supported model iff $T_{\mathcal{P}}(I) = I$.

The Core Method

- Let $\mathcal{L}$ be a logic language.
- Given a logic program $\mathcal{P}$ together with immediate consequence operator $T_\mathcal{P}$.
- Let $\mathcal{I}$ be the set of interpretations for $\mathcal{P}$.
- Find a mapping $R : \mathcal{I} \longrightarrow \mathbb{R}^n$.
- Construct a feed-forward network computing $f_\mathcal{P} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, called the core, such that the following holds:
  - If $T_\mathcal{P}(I) = J$ then $f_\mathcal{P}(R(I)) = R(J)$, where $I, J \in \mathcal{I}$.
  - If $f_\mathcal{P}(\vec{s}) = \vec{t}$ then $T_\mathcal{P}(R^{-1}(\vec{s})) = R^{-1}(\vec{t})$, where $\vec{s}, \vec{t} \in \mathbb{R}^n$.
- Connect the units in the output layer recursively to the units in the input layer.
- Show that the following holds
  - $I = \mathit{lfp}\,(T_\mathcal{P})$ iff the recurrent network converges to or approximates $R(I)$.

Connectionist model generation using recurrent networks with feed forward core.

3-Layer Recurrent Networks



- At each point in time all units do:
  - apply activation function to obtain potential,
  - apply output function to obtain output.

Propositional Core Method using Binary Threshold Units

- **Let $\mathcal{L}$ be the language of propositional logic over a set $\mathcal{V}$ of variables.**
- **Let $\mathcal{P}$ be a propositional logic program, e.g.,**

$$\mathcal{P} = \{A,\ C \leftarrow A \wedge \neg B,\ C \leftarrow \neg A \wedge B\}.$$

- $\mathcal{I} = 2^{\mathcal{V}}$ **is the set of interpretations for $\mathcal{P}$.**
- $T_{\mathcal{P}}(I) = \{A \mid A \leftarrow L_1 \wedge \ldots \wedge L_m \in \mathcal{P}$ **such that** $I \models L_1 \wedge \ldots \wedge L_m\}$.

$$\begin{array}{lcl}
T_{\mathcal{P}}(\emptyset) & = & \{A\} \\
T_{\mathcal{P}}(\{A\}) & = & \{A, C\} \\
T_{\mathcal{P}}(\{A, C\}) & = & \{A, C\} \quad = \quad \mathit{lfp}\,(T_{\mathcal{P}})
\end{array}$$

Representing Interpretations

- $\mathcal{I} = 2^{\mathcal{V}}$
- Let $n = |\mathcal{V}|$ and identify $\mathcal{V}$ with $\{1, \ldots, n\}$.
- Define $R : \mathcal{I} \rightarrow \mathbb{R}^n$ such that for all $1 \leq j \leq n$ we find:

$$R(I)[j] = \begin{cases} 1 & \text{if } j \in I, \\ 0 & \text{if } j \notin I. \end{cases}$$

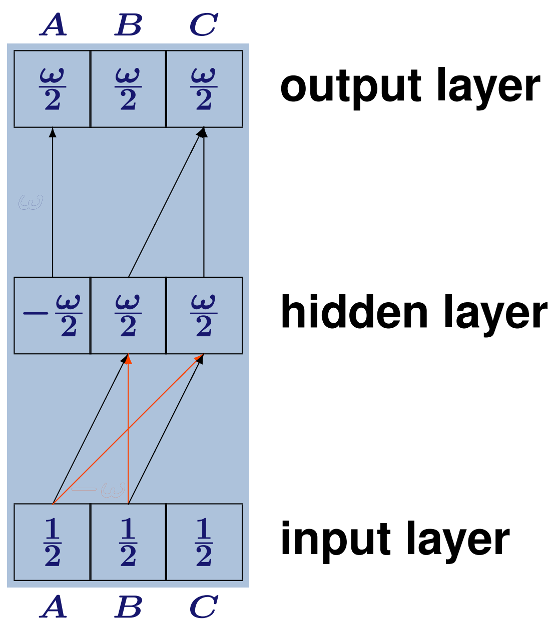
E.g., if $\mathcal{V} = \{A, B, C\} = \{1, 2, 3\}$ and $I = \{A, C\}$ then $R(I) = (1, 0, 1)$.

- Other encodings are possible, e.g.,

$$R(I)[j] = \begin{cases} 1 & \text{if } j \in I, \\ -1 & \text{if } j \notin I. \end{cases}$$

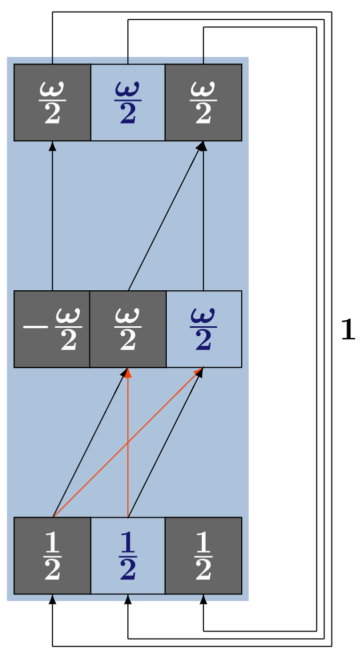
Computing the Core

- Consider again $\mathcal{P} = \{A,\ C \leftarrow A \wedge \neg B,\ C \leftarrow \neg A \wedge B\}$.
- A translation algorithm translates $\mathcal{P}$ into a core of binary threshold units:



Some Results

- **Proposition** 2-layer networks cannot compute $T_{\mathcal{P}}$ for definite $\mathcal{P}$.
- **Theorem** For each program $\mathcal{P}$, there exists a core computing $T_{\mathcal{P}}$.
- Recall $\mathcal{P} = \{A,\ C \leftarrow A \wedge \neg B,\ C \leftarrow \neg A \wedge B\}$.
- Adding recurrent connections:

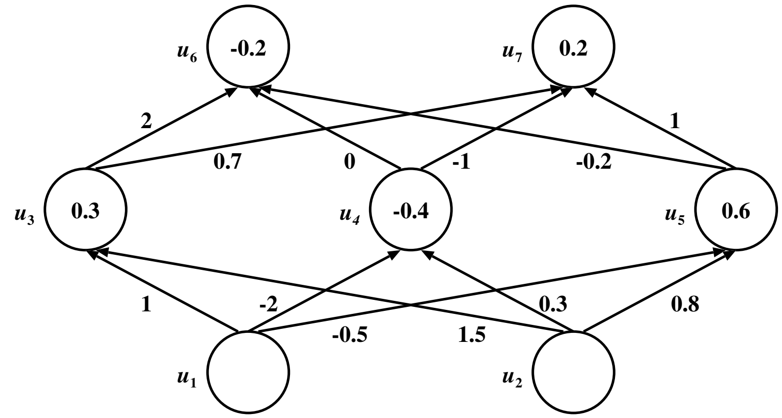


More Results

- A logic programs $\mathcal{P}$ is said to be strongly determined if there exists a metric d on the set of all Herbrand interpretations for $\mathcal{P}$ such that $T_{\mathcal{P}}$ is a contraction wrt d.
- Corollary Let $\mathcal{P}$ be a strongly determined program. Then there exists a core with recurrent connections such that the computation with an arbitrary initial input converges and yields the unique fixed point of $T_{\mathcal{P}}$.
- Let n be the number of clauses and m be the number of propositional variables occurring in $\mathcal{P}$.
  - $2m + n$ units, $2mn$ connections in the core.
  - $T_{\mathcal{P}}(I)$ is computed in 2 steps.
  - The parallel computational model to compute $T_{\mathcal{P}}(I)$ is optimal.
  - The recurrent network settles down in $3n$ steps in the worst case.
- See Hölldobler, Kalinke 1994 or Hitzler, Hölldobler, Seda 2004 for details.

Rule Extraction (1)

- **Proposition**
  For each core C there exists a program $\mathcal{P}$ such that C computes $T_{\mathcal{P}}$.



u_1	u_2	u_3		u_4		u_5		u_6		u_7	
		p_3	v_3	p_4	v_4	p_5	v_5	p_6	v_6	p_7	v_7
0	0	0	0	0	1	0	0	0	1	-1	0
0	1	1.5	1	.3	1	.8	1	1.8	1	.7	1
1	0	1	1	-1	0	$-.5$	0	2	1	.7	1
1	1	2.5	1	$-.7$	0	.3	0	2	1	.7	1

Rule Extraction (2)

- **Extracted program:**

$$
\begin{array}{lll}
P = \{ & A_1 \leftarrow \neg A_1 \wedge \neg A_2, & \\
& A_1 \leftarrow \neg A_1 \wedge A_2, & A_2 \leftarrow \neg A_1 \wedge A_2, \\
& A_1 \leftarrow A_1 \wedge \neg A_2, & A_2 \leftarrow A_1 \wedge \neg A_2, \\
& A_1 \leftarrow A_1 \wedge A_2, & A_2 \leftarrow A_1 \wedge A_2 \qquad \}.
\end{array}
$$

- **Simplified form:**

$$P = \{A_1,\ A_2 \leftarrow A_1,\ A_2 \leftarrow \neg A_1 \wedge A_2\}.$$

3-Layer Feed-Forward Networks Revisited

- **Theorem** (Funahashi 1989) Suppose that $\Psi : \mathbb{R} \longrightarrow \mathbb{R}$ is non-constant, bounded, monotone increasing and continuous. Let $K \subseteq \mathbb{R}^n$ be compact, let $f : K \longrightarrow \mathbb{R}$ be continuous, and let $\varepsilon > 0$. Then there exists a 3-layer feed-forward network with output function Ψ for the hidden layer and linear output function for the input and output layer whose input-output mapping $\overline{f} : K \longrightarrow \mathbb{R}$ satisfies

$$\max_{x \in K} |f(x) - \overline{f}(x)| < \varepsilon.$$

  - Every continuous function $f : K \longrightarrow \mathbb{R}$ can be uniformly approximated by input-output functions of 3-layer feed-forward networks.

- u_k is a sigmoidal unit if

$$\begin{array}{rcccl} \Phi(\vec{i_k}) & = & p_k & = & \sum_{j=1}^{m} w_{kj} v_j \\ \Psi(p_k) & = & v_k & = & \frac{1}{1+e^{\beta(\theta_k - p_k)}} \end{array}$$

where $\theta_k \in \mathbb{R}$ is a threshold (or bias) and $\beta > 0$ a steepness parameter.

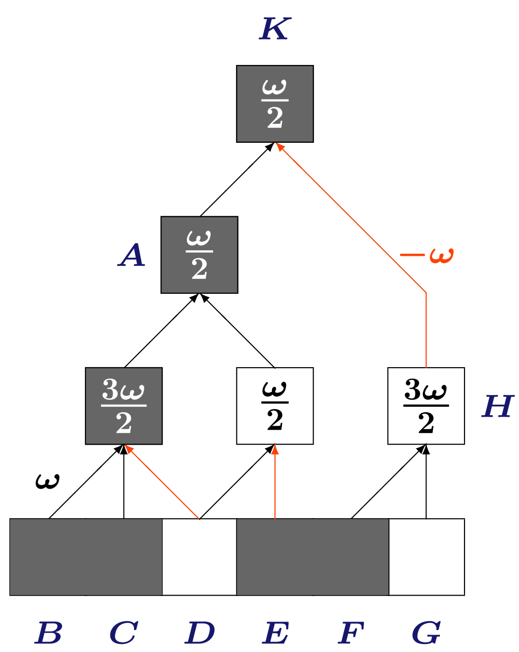
Backpropagation

- Bryson, Ho 1969, Werbos 1974, Parker 1985, Rumelhart, etal. 1986:
  Can 3-layer feed-forward networks learn a particular function?
- Training set of input-output pairs $\{(\vec{i^l}, \vec{o^l}) \mid 1 \leq l \leq n\}$.
- Minimize $E = \sum_l E^l$ where $E^l = \frac{1}{2} \sum_k (o_k^l - v_k^l)^2$.
- Gradient descent algorithm to learn appropriate weights.
- Backpropagation
  1. Initialize weights arbitrarily.
  2. Present input pattern $\vec{i^l}$ at time t.
  3. Compute output pattern $\vec{v^l}$ at time $t + 2$.
  4. Change weights according to $\Delta w_{ij}^l = \eta \delta_i^l v_j^l$, where
     - $\delta_i^l = \begin{cases} \Psi_i'(p_i^l) \times (o_i^l - v_i^l) & \text{if } i \text{ is output unit,} \\ \Psi_i'(p_i^l) \times \sum_k \delta_k^l w_{ki} & \text{if } i \text{ is hidden unit,} \end{cases}$
     - $\eta > 0$ is called learning rate.

Knowledge Based Artificial Neural Networks

- **Towell, Shavlik 1994:** Can we do better than empirical learning?
- Sets of hierarchical logic programs, e.g.,

$$\mathcal{P} = \{A \leftarrow B \wedge C \wedge \neg D,\ A \leftarrow D \wedge \neg E,\ H \leftarrow F \wedge G,\ K \leftarrow A, \neg H\}.$$

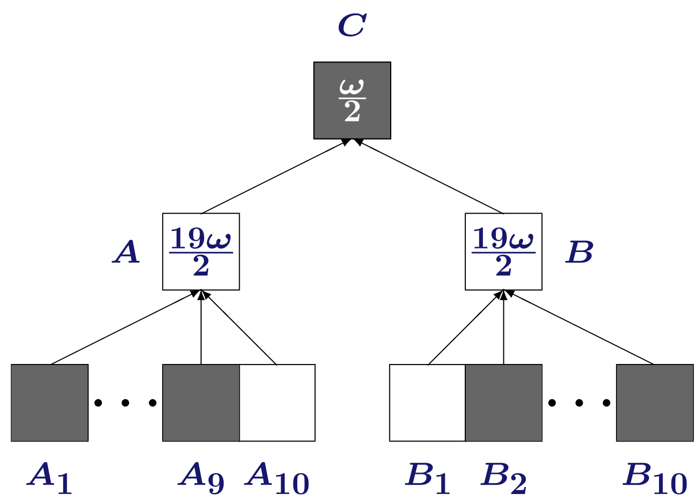


Knowledge Based Artificial Neural Networks – Learning

- Given hierachical sets of propositional rules as background knowledge.
- Map rules into multi-layer feed forward networks with sigmoidal units.
- Add hidden units (optional).
- Add units for known input features that are not referenced in the rules.
- Fully connect layers.
- Add near-zero random numbers to all links and thresholds.
- Apply backpropagation.
  - Empirical evaluation: system performs better than purely empirical and purely hand-built classifiers.

Knowledge Based Artificial Neural Networks – A Problem

- Works if rules have few conditions and there are few rules with the same head.



- $p_A = p_B = 9\omega$ and $v_A = v_B = \frac{1}{1+e^{\beta(9.5\omega - 9\omega)}} \approx 0.46$ with $\beta = 1$.
- $p_C = 0.92\omega$ and $v_c = \frac{1}{1+e^{\beta(0.5\omega - 0.92\omega)}} \approx 0.6$ with $\beta = 1$.

Propositional Core Method using Pipolar Sigmoidal Units

- d'Avila Garcez, Zaverucha, Carvalho 1997:
  Can we combine the ideas in Hölldobler, Kalinke 1994 and Towell, Shavlik 1994 while avoiding the above mentioned problem?
- Consider propositional logic language.
- Let I be an interpretation and $a \in [0, 1]$.

$$R(I)[j] = \begin{cases} v \in [a, 1] & \text{if } j \in I, \\ w \in [-1, -a] & \text{if } j \notin I. \end{cases}$$

- Replace threshold and sigmoidal units by bipolar sigmoidal ones, i.e., units with

$$\begin{array}{rclcl} \Phi(\vec{i}_k) & = & p_k & = & \sum_{j=1}^{m} w_{kj} v_j, \\ \Psi(p_k) & = & v_k & = & \frac{2}{1+e^{\beta(\theta_k - p_k)}} - 1, \end{array}$$

  where $\theta_k \in \mathbb{R}$ is a threshold (or bias) and $\beta > 0$ a steepness parameter.

The Task

- How should a, ω and θ_i be selected such that:
  - $v_i \in [a, 1]$ or $v_i \in [-1, -a]$ and
  - the core computes the immediate consequence operator?

Hidden Layer Units

- Consider $A \leftarrow L_1 \wedge \ldots \wedge L_n$.
- Let u be the hidden layer unit for this rule.
  - Suppose $I \models L_1 \wedge \ldots \wedge L_n$.
    - u receives input $\geq \omega a$ from unit representing L_i.
    - $p_u \geq n\omega a = p_u^+$.
  - Suppose $I \not\models L_1 \wedge \ldots \wedge L_n$.
    - u receives input $\leq -\omega a$ from at least one unit representing L_i.
    - $p_u \leq (n-1)\omega 1 - \omega a = p_u^-$.
- $\theta_u = \frac{n\omega a + (n-1)\omega - \omega a}{2} = (na + n - 1 - a)\frac{\omega}{2} = (n-1)(a+1)\frac{\omega}{2}$.

Output Layer Units

- Let μ be the number of clause with head A.
- Consider $A \leftarrow L_1 \wedge \ldots \wedge L_n$.
- Suppose $I \models L_1 \wedge \ldots \wedge L_n$.
  - $p_A \geq \omega a + (\mu - 1)\omega(-1) = \omega a - (\mu - 1)\omega = p_A^+$.
- Suppose for all rules of the form $A \leftarrow L_1 \wedge \ldots \wedge L_n$ we find $I \not\models L_1 \wedge \ldots \wedge L_n$.
  - $p_A \leq -\mu\omega a = p_A^-$.
- $\theta_A = \frac{\omega a - (\mu - 1)\omega - \mu\omega a}{2} = (a - \mu + 1 - \mu a)\frac{\omega}{2} = (1 - \mu)(a + 1)\frac{\omega}{2}$.

Computing a Value for a

- $p_u^+ > p_u^-$:
  - $n\omega a > (n-1)\omega - \omega a$.
  - $n\omega a + \omega a > (n-1)\omega$.
  - $a(n+1)\omega > (n-1)\omega$.
  - $a > \frac{n-1}{n+1}$.
- $p_A^+ > p_A^-$:
  - $\omega a - (\mu - 1)\omega > -\mu a \omega$.
  - $\omega a + \mu a \omega > (\mu - 1)\omega$.
  - $a(1+\mu)\omega > (\mu - 1)\omega$.
  - $a > \frac{\mu-1}{\mu+1}$.
- Consider all rules $\rightsquigarrow$ minimum value for a.

Computing a Value for ω

- $\Psi(p) = \frac{2}{1+e^{\beta(\theta-p)}} - 1 \geq a$.
- $\frac{2}{1+e^{\beta(\theta-p)}} \geq 1 + a$.
- $\frac{2}{1+a} \geq 1 + e^{\beta(\theta-p)}$.
- $\frac{2}{1+a} - 1 = \frac{2}{1+a} - \frac{1+a}{1+a} = \frac{1-a}{1+a} \geq e^{\beta(\theta-p)}$.
- $\ln(\frac{1-a}{1+a}) \geq \beta(\theta - p)$.
- $\frac{1}{\beta}\ln(\frac{1-a}{1+a}) \geq \theta - p$.
- **Consider a hidden layer unit:**
  - $\frac{1}{\beta}\ln(\frac{1-a}{1+a}) \geq (n-1)(a+1)\frac{\omega}{2} - n\omega a = \frac{na+n-a-1-2na}{2}\omega = \frac{n-1-a(n+1)}{2}\omega$.
  - $\omega \geq \frac{2}{(n-1-a(n+1))\beta}\ln(\frac{1-a}{1+a})$ **because** $a \geq \frac{n-1}{n+1}$.
- **Consider all hidden and output layer units as well as the case that** $\Psi(p) \leq -a$**:**
  - $\rightsquigarrow$ **minimum value for** ω.

Results

- Relation to logic programs is preserved.
- The core is trainable by backpropagation.
- Many interesting applications, e.g.:
  - DNA sequence analysis.
  - Power system fault diagnosis.
- Empirical evaluation:
  system performs better than well-known machine learning systems.
- See d'Avila Garcez, Broda, Gabbay 2002 for details.

Further Extensions

- Many-valued logic programs
- Modal logic programs
- Answer set programming
- Metalevel priorities
- Rule extraction

Propositional Core Method – Three-Valued Logic Programs

- **Kalinke 1994:** Consider truth values $\top,\ \bot,\ u$.
- Interpretations are pairs $I = \langle I^+, I^- \rangle$.
- Immediate consequence operator $\Phi_{\mathcal{P}}(I) = \langle J^+, J^- \rangle$, where

$$\begin{array}{rcl} J^+ & = & \{A \mid A \leftarrow L_1 \wedge \ldots \wedge L_m \in \mathcal{P} \text{ and } I(L_1 \wedge \ldots \wedge L_m) = \top\}, \\ J^- & = & \{A \mid \text{for all } A \leftarrow L_1 \wedge \ldots \wedge L_m \in \mathcal{P} :\ I(L_1 \wedge \ldots \wedge L_m) = \bot\}. \end{array}$$

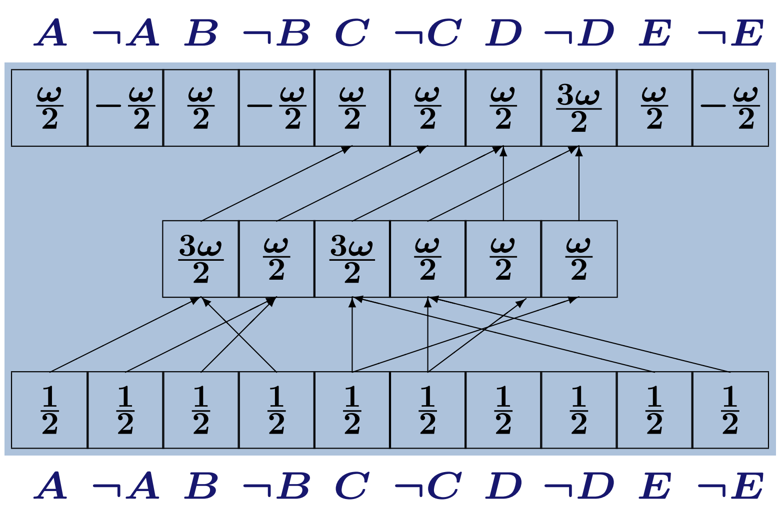
- Let $n = |\mathcal{V}|$ and identify $\mathcal{V}$ with $\{1, \ldots, n\}$.
- Define $R : \mathcal{I} \longrightarrow \mathbb{R}^{2n}$ as follows:

$$R(I)[2j-1] = \left\{ \begin{array}{ll} 1 & \text{if } j \in I^+ \\ 0 & \text{if } j \notin I^+ \end{array} \right\} \quad \text{and} \quad R(I)[2j] = \left\{ \begin{array}{ll} 1 & \text{if } j \in I^- \\ 0 & \text{if } j \notin I^- \end{array} \right\}$$

Propositional Core Method – Multi-Valued Logic Programs

- **For each program $\mathcal{P}$, there exists a core computing $\Phi_{\mathcal{P}}$, e.g.,**

$$\mathcal{P} = \{C \leftarrow A \wedge \neg B,\ D \leftarrow C \wedge E,\ D \leftarrow \neg C\}.$$



- **Lane, Seda 2004: Extension to finitely determined sets of truth values.**

Propositional Core Method – Modal Logic Programs

- **d'Avila Garcez, Lamb, Gabbay 2002.**
- **Let $\mathcal{L}$ be a propositional logic language plus**
  - **the modalities $\Box$ and $\Diamond$, and**
  - **a finite set of labels $w_1, \ldots, w_k$ denoting worlds.**
- **Let B be an atom, then $\Box B$ and $\Diamond B$ are modal atoms.**
- **A modal definite logic program $\mathcal{P}$ is a set of clauses of the form**

$$w_i : A \leftarrow A_1 \wedge \ldots \wedge A_m$$

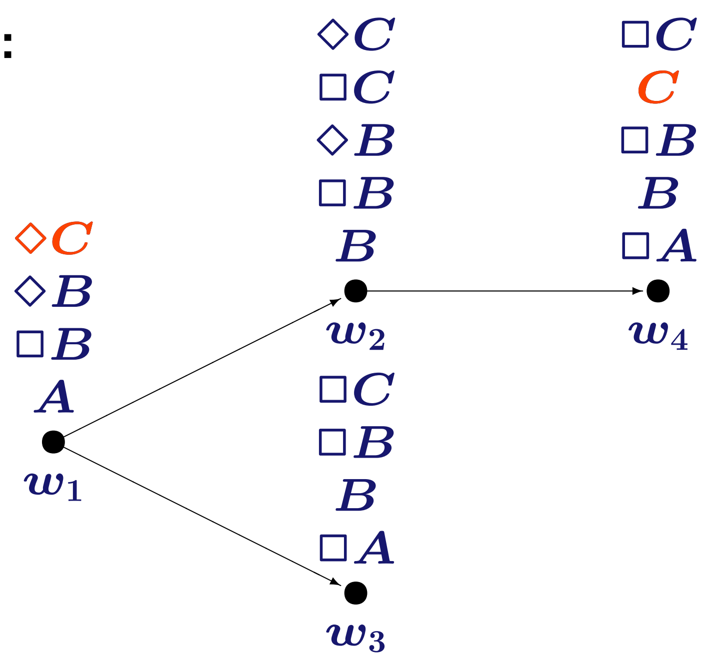
together with a finite set of relations $w_i \triangleright w_j$, where $w_i, 1 \leq i, j \leq k$, are labels and $A,\ A_1, \ldots, A_m$ are atoms or modal atoms.

- **$\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$, where $\mathcal{P}_i$ consists of all clauses labelled with w_i.**

Modal Logic Programs – Semantics

- **Example:**
$$
\begin{array}{rcl}
\mathcal{P} & = & \{w_1 : A,\ w_1 : \Diamond C \leftarrow A\} \\
& \cup & \{w_2 : B\} \\
& \cup & \{w_3 : B\} \\
& \cup & \{w_4 : B\} \\
& \cup & \{w_1 \blacktriangleright w_2,\ w_1 \blacktriangleright w_3,\ w_1 \blacktriangleright w_4,\ w_2 \blacktriangleright w_4,\}
\end{array}
$$

- **Kripke semantics:**



$$f_C(w_1) = w_4$$

Modal Immediate Consequence Operator

- Interpretations are tuples $I = \langle I_1, \ldots, I_k \rangle$
- Immediate consequence operator $MT_{\mathcal{P}}(I) = \langle J_1, \ldots, J_k \rangle$, where

$$
\begin{array}{rcll}
J_i & = & \{A & \mid \text{there exists } A \leftarrow A_1 \wedge \ldots \wedge A_m \in \mathcal{P}_i \\
 & & & \text{such that } \{A_1, \ldots, A_m\} \subseteq I_i\} \\
 & \cup & \{\Diamond A & \mid \text{there exists } w_i \blacktriangleright w_j \in \mathcal{P} \text{ and } A \in I_j\} \\
 & \cup & \{\Box A & \mid \text{for all } w_i \blacktriangleright w_j \in \mathcal{P} \text{ we find } A \in I_j\} \\
 & \cup & \{A & \mid \text{there exists } w_j \blacktriangleright w_i \in \mathcal{P} \text{ and } \Box A \in I_j\} \\
 & \cup & \{A & \mid \text{there exists } w_j \blacktriangleright w_i \in \mathcal{P},\ \Diamond A \in I_j \text{ and } f_A(w_j) = w_i\}
\end{array}
$$

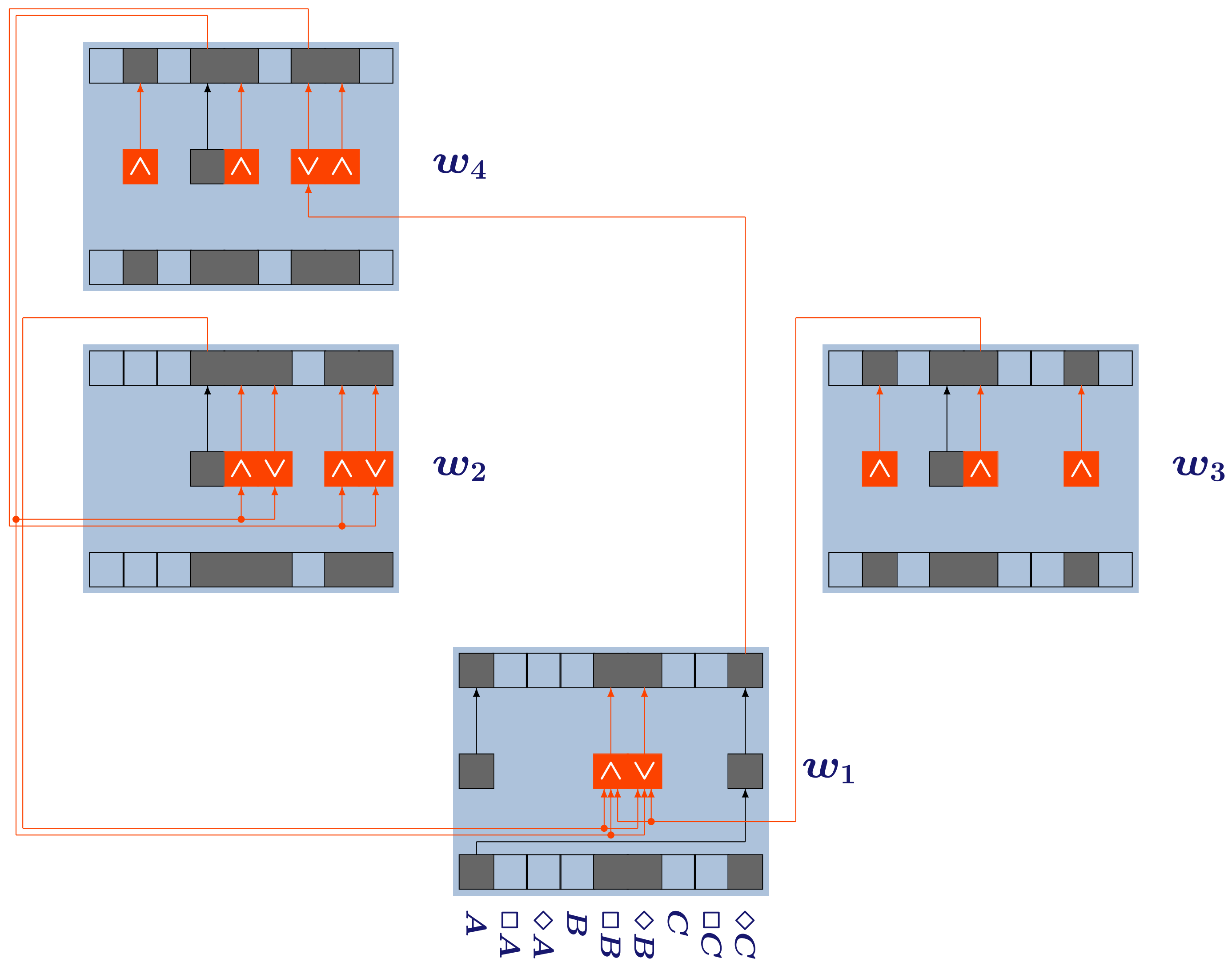
Modal Logic Programs – The Translation Algorithm

- Let $n = |\mathcal{V}|$ and identify $\mathcal{V}$ with $\{1, \ldots, n\}$.
- Let $a \in [0, 1]$.
- Define $R : \mathcal{I} \longrightarrow \mathbb{R}^{3n}$ as follows:

$$
\begin{aligned}
R(I)[3j-2] &= \begin{cases} v \in [a, 1] & \text{if } j \in I_j \\ w \in [-1, -a] & \text{if } j \not\in I_j \end{cases} \\
R(I)[3j-1] &= \begin{cases} v \in [a, 1] & \text{if } \Box j \in I_j \\ w \in [-1, -a] & \text{if } \Box j \not\in I_j \end{cases} \\
R(I)[3j] &= \begin{cases} v \in [a, 1] & \text{if } \Diamond j \in I_j \\ w \in [-1, -a] & \text{if } \Diamond j \not\in I_j \end{cases}
\end{aligned}
$$

- Translation algorithm such that
  - for each world the “local” part of $MT_{\mathcal{P}}$ is computed by a core,
  - the cores are turned into recurrent networks, and
  - the cores are connected with respect to the given set of relations.

The Example Network



First-Order Logic

- **Existing Approaches**
  - **Reflexive Reasoning and** SHRUTI
  - **Connectionist Term Representations**
    - **Holographic Reduced Representations Plate 1991**
    - **Recursive Auto-Associative Memory Pollack 1988**
  - **Horn logic and** CHCL **Hölldobler 1990, Hölldobler, Kurfess 1992**
  - **Other Approaches**
- **First-Order Logic Programs and the Core Method**
  - **Initial Approach**
  - **Construction of Approximating Networks**
  - **Topological Analysis and Generalisations**
  - **Employing Iterated Function Systems**

Reflexive Reasoning

- Humans are capable of performing a wide variety of cognitive tasks with extreme ease and efficiency.
- For traditional AI systems, the same problems turn out to be intractable.
- Human consensus knowledge: about 10^8 rules and facts.
- Wanted: "Reflexive" decisions within sublinear time.
- Shastri, Ajjanagadde 1993: SHRUTI.

SHRUTI – Knowledge Base

- Finite set of constants $\mathcal{C}$, finite set of variables $\mathcal{V}$.
- Rules:
  - $(\forall X_1 \ldots X_m)\,(p_1(\ldots) \wedge \ldots \wedge p_n(\ldots) \rightarrow (\exists Y_1 \ldots Y_k\, p(\ldots)))$.
  - $p,\ p_i, 1 \leq i \leq n$, are multi-place predicate symbols.
  - Arguments of the p_i: variables from $\{X_1, \ldots, X_m\} \subseteq \mathcal{V}$.
  - Arguments of p are from $\{X_1, \ldots, X_m\} \cup \{Y_1, \ldots, Y_k\} \cup \mathcal{C}$.
  - $\{Y_1, \ldots, Y_k\} \subseteq \mathcal{V}$.
  - $\{X_1, \ldots, X_m\} \cap \{Y_1, \ldots, Y_k\} = \emptyset$.
- Facts and queries (goals):
  - $(\exists Z_1 \ldots Z_l)\, q(\ldots)$.
  - Multi-place predicate symbol q.
  - Arguments of q are from $\{Z_1, \ldots, Z_l\} \cup \mathcal{C}$.
  - $\{Z_1, \ldots, Z_l\} \subseteq \mathcal{V}$.

Further Restrictions

- Restrictions to rules, facts, and goals:
  - No function symbols except constants.
  - Only universally bound variables may occur as arguments in the conditions of a rule.
  - All variables occurring in a fact or goal occur only once and are existentially bound.
  - An existentially quantified variable is only unified with variables.
  - A variable which occurs more than once in the conditions of a rule must occur in the conclusion of the rule and must be bound when the conclusion is unified with a goal.
  - A rule is used only a fixed number of times.

$\rightsquigarrow$ Incompleteness.

SHRUTI – **Example**

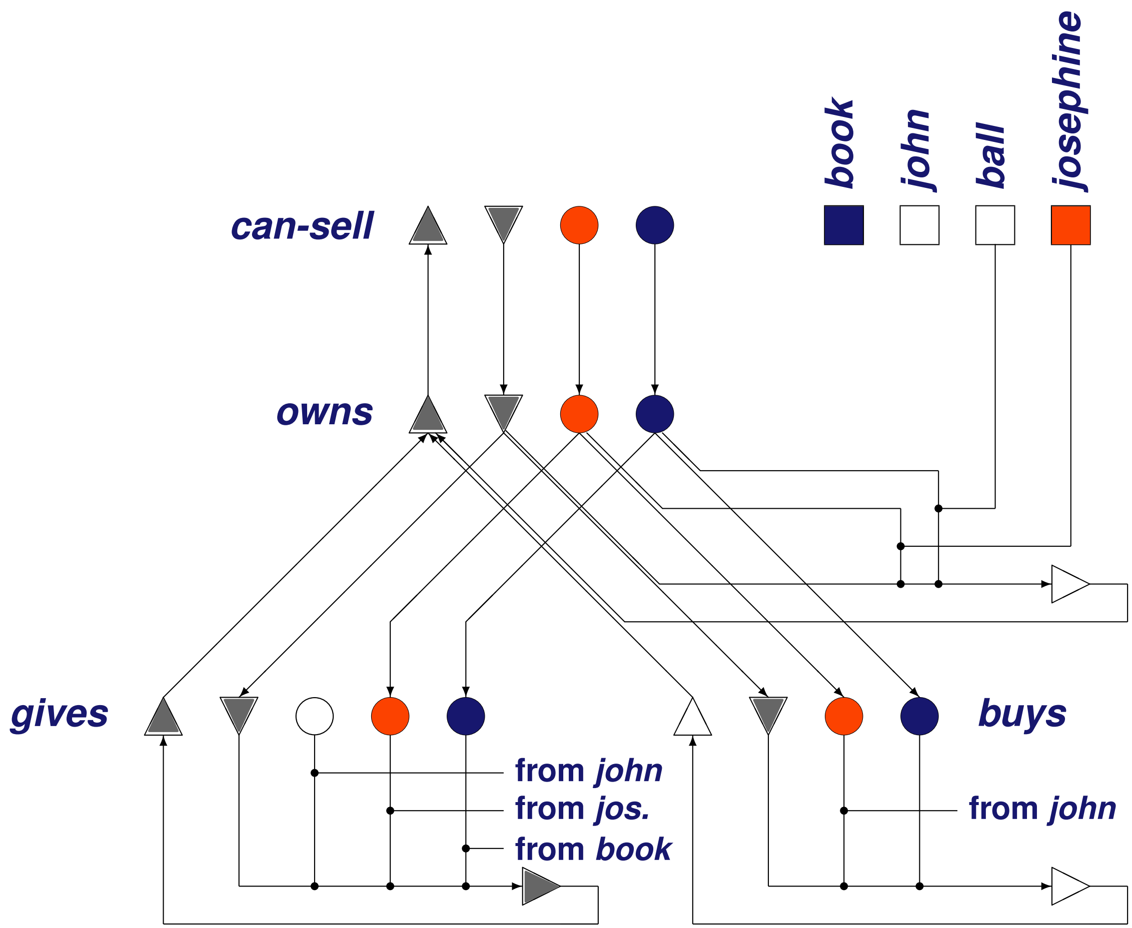
- **Rules**

$$\mathcal{P} = \{ \begin{array}{l} \textit{owns}(Y, Z) \leftarrow \textit{gives}(X, Y, Z), \\ \textit{owns}(X, Y) \leftarrow \textit{buys}(X, Y), \\ \textit{can-sell}(X, Y) \leftarrow \textit{owns}(X, Y), \\ \textit{gives}(\textit{john}, \textit{josephine}, \textit{book}), \\ (\exists X)\, \textit{buys}(\textit{john}, X), \\ \textit{owns}(\textit{josephine}, \textit{ball}) \end{array} \},$$

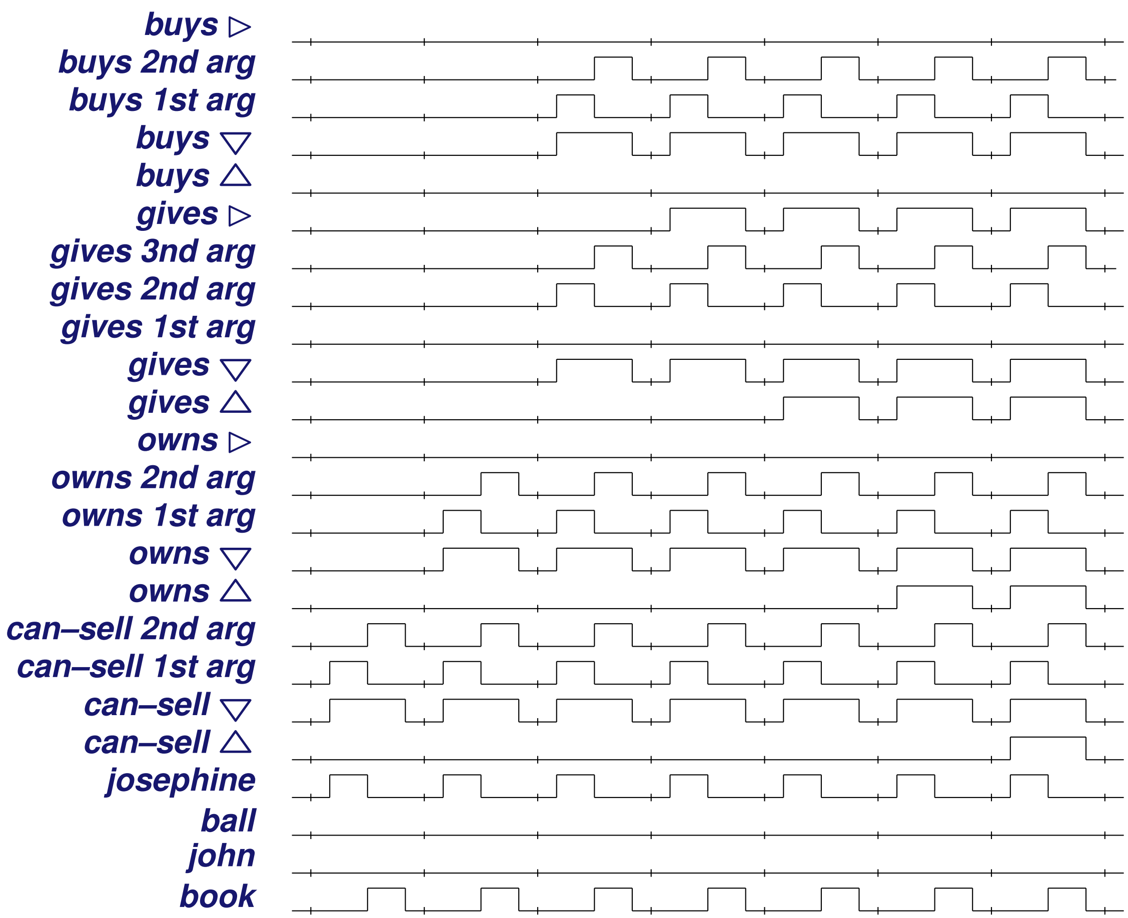
- **Queries:**

$$\begin{array}{lll} \textit{can-sell}(\textit{josephine}, \textit{book}) & \leadsto & \text{yes} \\ (\exists X)\, \textit{owns}(\textit{josephine}, X) & \leadsto & \text{yes} \quad \{X \mapsto \textit{book}\} \\ & & \phantom{\text{yes}} \quad \{X \mapsto \textit{ball}\} \end{array}$$

SHRUTI : The Network



Solving the Variable Binding Problem



SHRUTI – Remarks

- Answers are derived in time proportional to depth of search space.
- Number of units as well as of connections is linear in the size of the knowledge base.
- Extensions:
  - compute answer substitutions
  - allow a fixed number of copies of rules
  - allow multiple literals in the body of a rule
  - built in a taxonomy
- ROBIN (Lange, Dyer 1989): signatures instead of phases.
- Biological plausibility.
- Trading expressiveness for time and size.
- Logical reconstruction by Beringer, Hölldobler 1993:
  - Reflexive reasoning is reasoning by reduction.

First-Order Logic Programs and the Core Method

- Initial Approach
- Construction of Approximating Networks
- Topological Analysis and Generalisations
- Employing Iterated Function Systems

Logic Programs

- A logic program $\mathcal{P}$ over a first-order language $\mathcal{L}$ is a finite set of clauses

$$A \leftarrow L_1 \wedge \ldots \wedge L_n,$$

where A is an atom, L_i are literals and $n \geq 0$.

- $B_{\mathcal{L}}$ is the set of all ground atoms over $\mathcal{L}$ called Herbrand base.
- A Herbrand interpretation I is a mapping $B_{\mathcal{L}} \longrightarrow \{\top, \bot\}$.
- $2^{B_{\mathcal{L}}}$ is the set of all Herbrand interpretations.
- ground$(\mathcal{P})$ is the set of all ground instances of clauses in $\mathcal{P}$.
- Immediate consequence operator $T_{\mathcal{P}} : 2^{B_{\mathcal{L}}} \longrightarrow 2^{B_{\mathcal{L}}}$:

$$T_{\mathcal{P}}(I) = \{A \mid \text{there is a clause } A \leftarrow L_1 \wedge \ldots \wedge L_n \in \text{ground}(\mathcal{P}) \text{ such that } I \models L_1 \wedge \ldots \wedge L_n\}.$$

- I is a supported model iff $T_{\mathcal{P}}(I) = I$.

The Initial Approach

- **Hölldobler, Kalinke, Störr 1999:**
  Can the core method be extended to first-order logic programs?
- **Problem**
  - **Given a logic program $\mathcal{P}$ over a first order language $\mathcal{L}$ together with $T_{\mathcal{P}} : 2^{B_{\mathcal{L}}} \longrightarrow 2^{B_{\mathcal{L}}}$.**
  - **$B_{\mathcal{L}}$ is countably infinite.**
  - **The method used to relate propositional logic and connectionist systems is not applicable.**
  - **How can the gap between the discrete, symbolic setting of logic, and the continuous, real valued setting of connectionist networks be closed?**

The Goal

- Find $R : 2^{B_\mathcal{L}} \longrightarrow \mathbb{R}$ and $f_P : \mathbb{R} \longrightarrow \mathbb{R}$ such that the following conditions hold.
  - $T_\mathcal{P}(I) = I'$ implies $f_\mathcal{P}(R(I)) = R(I')$.
    $f_\mathcal{P}(x) = x'$ implies $T_\mathcal{P}(R^{-1}(x)) = R^{-1}(x')$.
    - $\rightsquigarrow$ $f_\mathcal{P}$ is a sound and complete encoding of $T_\mathcal{P}$.
  - $T_\mathcal{P}$ is a contraction on $2^{B_\mathcal{L}}$ iff $f_\mathcal{P}$ is a contraction on $\mathbb{R}$.
    - $\rightsquigarrow$ The contraction property and fixed points are preserved.
  - $f_\mathcal{P}$ is continuous on $\mathbb{R}$.
    - $\rightsquigarrow$ A connectionst network approximating $f_\mathcal{P}$ is known to exist.

Acyclic Logic Programs

- Let $\mathcal{P}$ be a program over a first order language $\mathcal{L}$.
- A level mapping for $\mathcal{P}$ is a function $l : B_{\mathcal{L}} \to \mathbb{N}$.
  - We define $l(\neg A) = l(A)$.
- We can associate a metric $d_{\mathcal{L}}$ with $\mathcal{L}$ and l. Let $I, J \in 2^{B_{\mathcal{L}}}$:

$$d_{\mathcal{L}}(I, J) = \begin{cases} 0 & \text{if } I = J \\ 2^{-n} & \text{if } n \text{ is the smallest level on which } I \text{ and } J \text{ differ.} \end{cases}$$

- Proposition (Fitting 1994) $(2^{B_{\mathcal{L}}}, d_{\mathcal{L}})$ is a complete metric space.
- $\mathcal{P}$ is said to be acyclic wrt a level mapping l, if for every $A \leftarrow L_1 \wedge \ldots \wedge L_n \in \text{ground}(\mathcal{P})$ we find $l(A) > l(L_i)$ for all i.
- Proposition Let $\mathcal{P}$ be an acyclic logic program wrt l and $d_{\mathcal{L}}$ the metric associated with $\mathcal{L}$ and l, then $T_{\mathcal{P}}$ is a contraction on $(2^{B_{\mathcal{L}}}, d_{\mathcal{L}})$.

Mapping Interpretations to Real Numbers

- Let $\mathcal{D} = \{r \in \mathbb{R} \mid r = \sum_{i=1}^{\infty} a_i 4^{-i},$ where $a_i \in \{0,1\}$ for all $i\}$.
- Let l be a bijective level mapping.
- $\{\top, \bot\}$ can be identified with $\{0,1\}$.
- The set of all mappings $B_{\mathcal{L}} \longrightarrow \{\top, \bot\}$ can be identified with the set of all mappings $\mathbb{N} \longrightarrow \{0,1\}$.
- Let $I_{\mathcal{L}}$ be the set of all mappings from $B_{\mathcal{L}}$ to $\{0,1\}$.
- Let $R : I_{\mathcal{L}} \longrightarrow \mathcal{D}$ be defined as

$$R(I) = \sum_{i=1}^{\infty} I(l^{-1}(i))4^{-i}.$$

- **Proposition** R is a bijection.

We have a sound and complete encoding of interpretations.

Mapping Immediate Consequence Operators to Functions on the Reals

- We define $f_{\mathcal{P}} : \mathcal{D} \longrightarrow \mathcal{D} : r \mapsto R(T_{\mathcal{P}}(R^{-1}(r)))$.

$$
\begin{tikzcd}
I \arrow[r, "T_{\mathcal{P}}"] \arrow[d, "R"'] & I' \arrow[d, "R"] \\
r \arrow[r, "f_{\mathcal{P}}"'] & r'
\end{tikzcd}
$$

We have a sound and complete encoding of $T_{\mathcal{P}}$.

- Proposition Let $\mathcal{P}$ be an acylic program wrt a bijective level mapping. $f_{\mathcal{P}}$ is a contraction on $\mathcal{D}$.

Contraction property and fixed points are preserved.

Approximating Continuous Functions

- **Corollary** $f_{\mathcal{P}}$ is continuous.
- Recall Funahashi's theorem:
  - Every continuous function $f : K \longrightarrow \mathbb{R}$ can be uniformly approximated by input-output functions of 3-layer feed forward networks.
- **Theorem** $f_{\mathcal{P}}$ can be uniformly approximated by input-output functions of 3-layer feed forward networks.
  - $T_{\mathcal{P}}$ can be approximated as well by applying R^{-1} .

Connectionist network approximating immediate consequence operator exists.

An Example

- Consider $P = \{q(0),\ q(s(X)) \leftarrow q(X)\}$ and let $l(q(s^n(0))) = n + 1$.
  - $\mathcal{P}$ is acyclic wrt l, l is bijective, $R(B_{\mathcal{L}}) = \frac{1}{3}$.
  - $$\begin{aligned} f_{\mathcal{P}}(R(I)) &= 4^{-l(q(0))} + \textstyle\sum_{q(X) \in I} 4^{-l(q(s(X)))} \\ &= 4^{-l(q(0))} + \textstyle\sum_{q(X) \in I} 4^{-(l(q(X))+1)} = \frac{1+R(I)}{4}. \end{aligned}$$
- Approximation of $f_{\mathcal{P}}$ to accuracy ε yields

$$\overline{f}(x) \in \left[\frac{1+x}{4} - \varepsilon, \frac{1+x}{4} + \varepsilon\right].$$

- Starting with some x and iterating $\overline{f}$ yields in the limit a value

$$r \in \left[\frac{1-4\varepsilon}{3}, \frac{1+4\varepsilon}{3}\right].$$

- Applying R^{-1} to r we find

$$q(s^n(0)) \in R^{-1}(r) \text{ if } n < -\log_4 \varepsilon - 1.$$

Approximation of Interpretations

- Let $\mathcal{P}$ be a logic program over a first order language $\mathcal{L}$ and l a level mapping.
- An interpretation I approximates an interpretation J to a degree $n \in \mathbb{N}$ if for all atoms $A \in B_{\mathcal{L}}$ with $l(A) < n$ we find $I(A) = \top$ iff $J(A) = \top$.
  - I approximates J to a degree n iff $d_{\mathcal{L}}(I, J) \leq 2^{-n}$.

Approximation of Supported Models

- Given an acyclic logic program $\mathcal{P}$ with bijective level mapping.
- Let $T_{\mathcal{P}}$ be the immediate consequence operator associated with $\mathcal{P}$ and $M_{\mathcal{P}}$ the least supported model of $\mathcal{P}$.
- We can approximate $T_{\mathcal{P}}$ by a 3-layer feed forward network.
- We can turn this network into a recurrent one.

Does the recurrent network approximate the supported model of $\mathcal{P}$?

- **Theorem** For an arbitrary $m \in \mathbb{N}$ there exists a recursive network with sigmoidal activation function for the hidden layer units and linear activation functions for the input and output layer units computing a function $\overline{f}_{\mathcal{P}}$ such that there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x \in [-1, 1]$ we find

$$d_{\mathcal{L}}(R^{-1}(\overline{f}^{n}_{\mathcal{P}}(x)), M_{\mathcal{P}}) \leq 2^{-m}.$$

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